Notes on Eilenberg and Kelly*

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December 1, 2020

必也正名乎 — 孔子
Most imperative is to rectify names.
— Confucius[12, XIII (Zi Lu) §3]

And summed it so well that it came to far more
Than the Witnesses ever had said!
— Lewis Carroll [9]

These are notes and commentary on the paper "*A generalization of the Functorial Calculus*" by Samual Eilenberg & G. Max Kelly [14], and some successor papers. (Actually, it is only part of my notes. Some parts are in the table of contents, but not included here.)

I was reading some (relatively) recent papers [10, 11, 8] which use Kelly-Mac Lane graphs and reference Kelly and Mac Lane [20]. That paper was relatively easy to get as a PDF file. It references Eilenberg and Kelly [14], which was more difficult to locate. After a quest and a saga, and some expert help from WPI librarian Judy Fallon, I was able to get a copy from a microfilm in the basement of the library.

Because the paper may be difficult to obtain, I want to write these notes in such detail that if one only wants to understand the ideas, there will be no need to read the original paper. Of course, if you are interested in the history of the subject, there is no substitute for the original.

The inititials E&K will be used for the paper [14] or its authors, depending upon context. I can't just re-type the paper, because that would violate the copyright, but I do go through point-by-point, explaining each paragraph, leaving nothing out. I do not duplicate the formulae of E&K but instead use my own notational conventions, as described in section 1. To aid comparison and avoid mistakes, I mostly use the same letters as variables, possibly changing the font or case. Discussion of the Kelly and Mac Lane [20] paper *Coherence in Closed Categories* is omitted from this abridged version, though it is in the Table of Contents.

Throughout, CWM is the canonical reference, namely Mac Lane's *Categories for the Working Mathematician* [24] assumed to be accessible to everyone.

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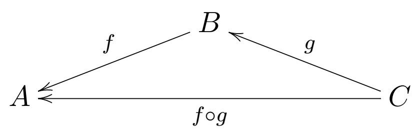


Figure 1: Composition

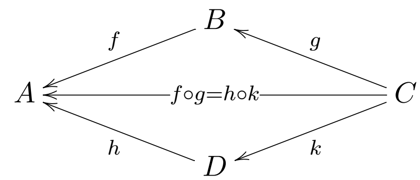


Figure 2: Equality

1 Notation and Terminology

The apostrophe in "$R‘y$" may be read "of".
— Whitehead and Russell[25, *30]

The first notational problem anyone writing about category theory faces is whether composition should be written left-to-right or right-to-left. On the one hand, right-to-left composition has a long history in all branches of mathematics, but left-to-right composition seems more consistent with the diagrams. I want something that works for everyone, and I don't want to be the one to tell all the sophmores and engineers in the world, that, despite what all the books say, from now on we write ${}^{ix}e = (x)\cos + (x)\sin i$. We will stay with the long tradition of right-to-left composition, but make the formulas match the diagrams by reversing the diagrams, The arrows in a diagram point left whenever possible, as shown in Figures 1 and 2.

Following Whitehead and Russell [25], an apostrophe symbolizes the operation of application of a function (or functor) e.g. $\cos‘\pi = -1$. A small circle symbolizes composition. Thus: $(f \circ g)‘x = f‘(g‘x)$. Either of those might be omitted, but we try to avoid omitting two different symbols in the same formula. Sometimes the apostrophe is omitted and redundant parenthesis are added: $\cos(\pi)$. Also following Whitehead and Russell $f“S = \{f‘x | x \in S\}$. Following Barendregt [6, §2.1.18], sometimes $[\,]$ is a hole into which a parameter can be placed.

In addition, when discussing an abstract category use: • Lower case letters for arrows; • Uppercase letters for objects (or identity arrows); • Roman letters for objects and arrows; • Bold letters for categories and functors; • Greek letters for natural transformations; • $[]$ is an identity arrrow, 1 is a terminal object.

The "hom" functor for a category $\mathbf{C}$ is $\mathbf{C}\{[\,] \leftarrow [\,]\} : \mathbf{Set} \longleftarrow \mathbf{C} \times \mathbf{C}^{op}$. The name of the category $\mathbf{C}$ can be omitted if it is understood. If $A, B \in \mathbf{C}$, then $\mathbf{C}\{A \leftarrow B\} = \{f | f : A \longleftarrow B \in \mathbf{C}\}$. As a functor, hom also applies to arrows: given two arrows $a : A \longleftarrow A'$ and $b : B' \longleftarrow B$, we define $\{a \leftarrow b\} : \{A \leftarrow B\} \longleftarrow \{A' \leftarrow B'\}$ by $\{a \leftarrow b\}‘f = a \circ f \circ b \in \{A \leftarrow B\}$ for every $f \in \{A' \leftarrow B'\}$.

Sometimes the identity on an object has the same name as the object. If $A : A \longleftarrow A$ and $B : B \longleftarrow B$, then $\{A \leftarrow B\}$ maps any $f \in \{A \leftarrow B\}$ to $A \circ f \circ B = f \in \{A \leftarrow B\}$. Thus, $\{A \leftarrow B\}$ is the identity function on the set of arrows of the same name.

For $a : A \longleftarrow A'$, $b : B' \longleftarrow B$, and $a' : A' \longleftarrow A''$, $b' : B'' \longleftarrow B'$, we have

$$\{a \leftarrow b\} : \{A \leftarrow B\} \longleftarrow \{A' \leftarrow B'\}, \qquad \{a' \leftarrow b'\} : \{A' \leftarrow B'\} \longleftarrow \{A'' \leftarrow B''\}$$

and $$\{a \leftarrow b\} \circ \{a' \leftarrow b'\} = \{(a \circ a') \leftarrow (b' \circ b)\} : \{A \leftarrow B\} \longleftarrow \{A'' \leftarrow B''\}$$

2 Eilenberg and Kelly: "*A Generalization of the Functorial Calculus*"

E&K begins by citing Godement [2, appendice[*sic*[1]]] for the "five rules of functorial calculus", and saying that they can be trivially generalized by applying them to product and opposite categories, but then complains that there are "useful situations which are not covered by the existing notions." Since the citation gives no publisher, but only a city (Paris), and in any case I do not read French, I did not look up that reference, but Barr and Wells [7, p19] give a list of five rules that they call "Godement's rules". These rules are discussed in subsection 4.1 of these notes. One might expect to see a sixth rule in E&K, but it does not seem to be there. Rather than derive new rules they define a new notion to cover a new situation.

For an example of a useful situation not covered by Godement's rules, let $\mathbf{Ab}$ be the category of Abelian groups[2], and consider the family of evaluation functions (group homomorphisms) indexed by pairs of Abelian groups $A, B \in \mathbf{Ab}$:

$$\epsilon(AB)\,\mathbf{:}\,B \longleftarrow A \otimes \mathbf{Ab}(AB)$$

The evaluation functions are given by $\epsilon_{AB}`(a \otimes f) = f`a \in B$ for $a \in A$ and $f \in \mathbf{Ab}(AB)$.

Here $f`a$ means the function f applied to the argument a. E&K use "a" for the evaluation functions, but one line later they also use "a" (as above) for an arbitrary element of A. I have taken the liberty of changing the "a" to "ϵ" when it denotes the evaluation function. I follow E&K in placing the indices in parentheses when discussing categorial typing in general, but moving them to subscripts when they are simple parts of a more complicated expression, as when the indexed function is to be applied to an argument.

E&K say $\mathbf{Ab}(AB)$ is an abbreviation for $\hom(A, B)$, but notice that $\hom(A, B) \in \mathbf{Ab}$, i.e. $\hom(A, B)$, the set of homomorphisms from A to B, forms an Abelian group. If that were not so, the tensor product with $\mathbf{Ab}(AB)$ could not be defined. As noted in the footnote to the above, E&K write $\mathcal{A}$ for $\mathbf{Ab}$ in these two paragraphs. Through the rest of the E&K paper, $\mathcal{A}$ is an arbitrary category, $\mathcal{A}(AB)$ is just a set of arrows, and $\mathcal{A}(AB) \in \mathcal{A}$ does not hold in general.

For those who are not familiar with tensor products of Abelian groups, the evaluation functor, E_X, for sets, functions, and cartesian products of sets, which is described by Mac Lane [24, CWM IX§4] may be easier to understand.

Returning to the E&K example, for each fixed A, the expression $A \otimes \mathbf{Ab}(AB)$ gives a functor of B, and $\epsilon(AB)$ a family of arrows, natural in B.

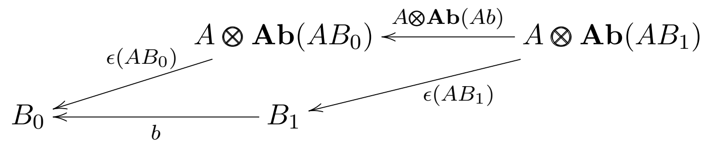
The words "natural in B" are a kind of variable binding. similar to the "λ" in lambda calculus. The letter B is an arbitrary parameter, and the expression in which it occurs is interpreted as a functor of that parameter. It does not make sense to ask for the value of B—it is an unspecified object in the category $\mathbf{Ab}$, i.e. it is any Abelian group. Furthermore, since the expression is a *functor* of the parameter, the parameter is not required to be an object, but could also be an arrow. So the functor maps the (abelian) group B to the group $A \otimes \mathbf{Ab}(AB)$ and also maps a group homomorphism $b\,\mathbf{:}\,B_0 \longleftarrow B_1$ to a homomorphism $h\,\mathbf{:}\,A \otimes \mathbf{Ab}(AB_0) \longleftarrow A \otimes \mathbf{Ab}(AB_1)$.

What is that h? Obviously $h = A \otimes \mathbf{Ab}(A\ b)$, but that seems like begging the question.

[1] It's french.

[2] Actually, E&K call this category $\mathcal{A}$. I follow Mac Lane [24, CWM] in calling it $\mathbf{Ab}$.

That $\epsilon(AB)$ is natural in B means that, for any fixed $A \in \mathbf{Ab}$, and any morphism $b\colon B_0 \longleftarrow B_1$, we have $\epsilon(AB_0) \circ (A \otimes \mathbf{Ab}(A\ b)) = b \circ \epsilon(AB_1)$.



On the other hand, for fixed B, the expression $A \otimes \mathbf{Ab}(AB)$ is not even a functor of A, and the question of naturality of $\epsilon(AB)$ can not even arise. Nevertheless, if we separate the two occurrences of A, for fixed B the expression $A_0 \otimes \mathbf{Ab}(A_1B)$ gives a bivalent functor[3] covariant in A_0 and contravariant in A_1. That is, $A_0 \otimes \mathbf{Ab}(A_1B)\colon \mathbf{Ab} \longleftarrow \mathbf{Ab} \times \mathbf{Ab}^{op} \times \mathbf{Ab}$ is a functor of A_0, A_1, and B, contravariant in A_1 and in some sense ϵ should be a natural transformation to the identity.

(Because the target of $\epsilon(AB)\colon B \longleftarrow A \otimes \mathbf{Ab}(AB)$ as a functor of B, is the identity on $\mathbf{B}$.)

It can not be a natural transformation in anything like the ordinary sense, because the two functors it relates are not even parallel. The rest of the paper develops a generalized sense of natural transformation called “extra-natural transformation” which has been designed to cover this case.

Note that, though it is traditional to use the same sign “$\otimes$” for both the bivalent functor “tensor product of two groups” and for the bivalent operation that takes a pair of elements of those groups to an element of the tensor product, and this is the notation used by E&K, I am not sure that is a good idea in general, and so I have used a bold faced $\otimes$ for the functor and a small $\otimes$ for the operation. Thus, $A \otimes \mathbf{Ab}(AB)$ is a group, $a{\otimes}f$ is a typical element of that group.

After this introductory paragraph E&K say that their generalization of the notion of a natural transformation was designed to include the above example, but then they never again mention or use tensor products. All products that follow are Cartesian products of categories.

2.1 Definitions of Transformations

E&K say that the basic definitions were given by Kelly [3], but that paper did not consider the composability of such “extraordinary natural transformations”. But such transformations are only useful if they can be composed. Whether two such transformations can be composed to get a third is a subtle question and the bulk of E&K is devoted to it.

The phrase “extraordinary natural transformations” is used by the original paper by Eilenberg and Kelly [14], which introduced the concept, but seems not to have become standard. Mac Lane [24, CWM IX§4] mentions the words “dinatural”, “extranatural”, “supernatural”, and “wedge”[4], for various special cases. We have no use for unnatural transformations, so let’s call them all simply “transformations”, reserving the the adjective “natural” for the ordinary case of a natural transformation between parallel functors as defined in most any introduction to category theory (including CWM [24, p.16,§I.4]).

Natural, wedge, and co-wedge transformations, are exhibited, but not separately named, by Eilenberg and Kelly. They say “the family $\alpha = \{\alpha(ABC)\}$ is *natural* if” the following definition applies:

[3] That is, a functor with two parameters. Some call this a bifunctor, but I say bifunctors are for bicategories.

[4] I once said: “CWM calls them both ‘wedges’. There is no ‘cowedge’ perhaps for fear of cows with edges.”, but that comment seems to have been made obsolete.

Definition 1. *Suppose given two functors*

$$\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B} \qquad \textit{and} \qquad \mathbf{s} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C}$$

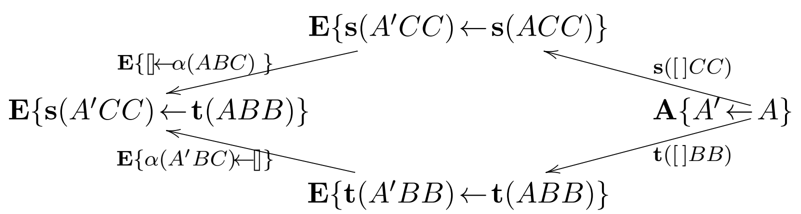
and a family of arrows in $\mathbf{E}$ *indexed by objects* $A \in \mathbf{A}$*,* $B \in \mathbf{B}$*,* $C \in \mathbf{C}$

$$\Big(\alpha(ABC) : \mathbf{s}(ACC) \longleftarrow \mathbf{t}(ABB)\Big)_{A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}}$$

This is a (natural or) extranatural family or is a transformation if the following three diagrams commute for all $A, A' \in \mathbf{A}$*, and* $B, B' \in \mathbf{B}$*, and* $C, C' \in \mathbf{C}$*.*

These first three diagrams are in the category of sets. In these diagrams, to save space, I use "[]" to stand for an identity arrow. Context will suffice to determine the object on which it is an identity.

Actually, the diagrams above are not only in the category of sets, but in the subcategory of sets of parallel arrows (hom sets). Arrows in the category of sets are functions, if these functions have hom sets for domain and codomain, then they may be functors, covariant if they preserve composition, contravariant if they reverse composition. Of course, if they do neither then they are not functors. All the functions in these diagrams are functors. The symbol ⥀ marks a contravariant functor. E&K did not mark the arrows, I did.

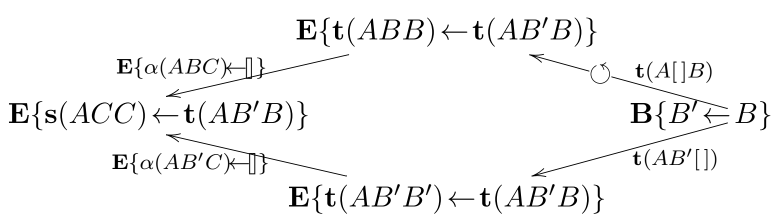


(1)

Natural transformation

$$\mathbf{E}\{[] \leftarrow \alpha(ABC)\} \circ \mathbf{s}([]CC) = \mathbf{E}\{\alpha(A'BC) \leftarrow []\} \circ \mathbf{t}([]BB)$$

In this diagram the "[]" in the target of the hom set function labeling the arrow in the upper left denotes the identity on $\mathbf{s}(A'CC)$, while that in the source of the lower left hom function denotes the identity on $\mathbf{t}(ABB)$. This can be seen by looking at the source and target of the labeled arrow.



(2)

Co-wedge transformation

$$\mathbf{E}\{\alpha(ABC) \leftarrow []\} \circ \mathbf{t}(A[]B) = \mathbf{E}\{\alpha(AB'C) \leftarrow []\} \circ \mathbf{t}(AB'[])$$

Here both occurrences of "[]" denote the identity on $\mathbf{t}(AB'B)$.

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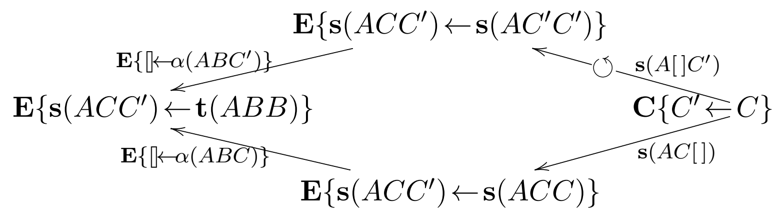
(3)



Wedge transformation

$$\mathbf{E}\{[]\leftarrow\alpha(ABC')\}\circ\mathbf{s}(A[]C')=\mathbf{E}\{[]\leftarrow\alpha(ABC)\}\circ\mathbf{s}(AC[])$$

Here "$[]$" denotes the identity on $\mathbf{s}(ACC')$.

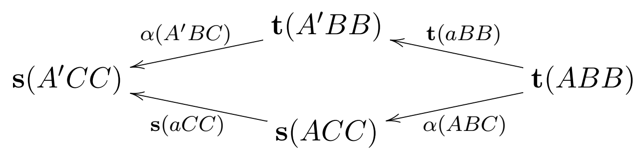
The assertion that α is a transformation to $\mathbf{s}$ from $\mathbf{t}$ can be symbolized as $\alpha:\mathbf{s}\Leftarrow\mathbf{t}$. This symbolic form is taken from Loregian [22] who may have taken it from Yoneda. An ordinary arrow would not be appropriate, because there is no category of transformations. We already have a symbolic notation for a natural transformation.

We need a short way to annotate the symbolic form with information which can be used to calculate whether $\beta:\mathbf{r}\Leftarrow\mathbf{s}$ can be composed with $\alpha:\mathbf{s}\Leftarrow\mathbf{t}$ to get $\beta\circ\alpha:\mathbf{r}\Leftarrow\mathbf{t}$.

The next three diagrams are in $\mathbf{E}$. When applied to $a: A'\longleftarrow A$, $b: B'\longleftarrow B$, and $c: C'\longleftarrow C$ in categories $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, respectively, each of the preceding three diagrams becomes the corresponding one of the following three diagrams. Each of the preceding diagrams commutes iff the corresponding following diagram commutes for all appropriate arrows, a, b, and c.

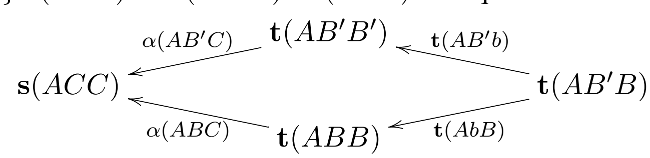
For example, to say that diagram (1) commutes is to say that for all $a\in\mathbf{A}\{A'\leftarrow A\}$ the two elements of $\mathbf{E}\{\mathbf{s}(A'CC)\leftarrow\mathbf{t}(ABB)\}$ given by $\mathbf{E}\{[]\leftarrow\alpha(ABC)\}`\mathbf{s}(aCC)=\mathbf{s}(aCC)\circ\alpha(ABC)$ and by $\mathbf{E}\{\alpha(A'BC)\leftarrow[]\}`\mathbf{t}(aBB)=\alpha(A'BC)\circ\mathbf{t}(aBB)$ are equal. This shown in diagram (1′).

(1′)



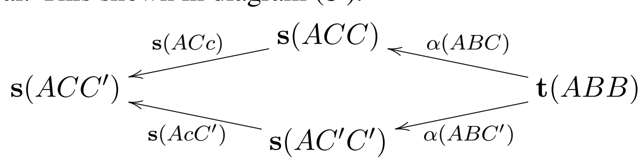
Similarly, diagram (2) commutes if for all $b\in\mathbf{B}\{B'\leftarrow B\}$ the two elements of $\mathbf{E}\{\mathbf{s}(ACC)\leftarrow\mathbf{t}(AB'B)\}$ given by $\mathbf{E}\{\alpha(ABC)\leftarrow[]\}`\mathbf{t}(AbC)=\alpha(ABC)\circ\mathbf{t}(AbC)$ and by $\mathbf{E}\{\alpha(AB'C)\leftarrow[]\}`\mathbf{t}(AB'b)=\alpha(A'BC)\circ\mathbf{t}(AB'b)$ are equal. This shown in diagram (2′).

(2′)



Finally, (3) commutes if $\forall c\in\mathbf{C}\{C'\leftarrow C\}$ the elements of $\mathbf{E}\{\mathbf{s}(ACC')\leftarrow\mathbf{t}(ABB)\}$ given by $\mathbf{E}\{[]\leftarrow\alpha(ABC)\}`\mathbf{s}(ACc)=\mathbf{s}(ACc)\circ\alpha(ABC)$ and by $\mathbf{E}\{[]\leftarrow\alpha(ABC')\}`\mathbf{s}(AcC')=\mathbf{s}(AcC')\circ\alpha(ABC')$ are equal. This shown in diagram (3′).

(3′)



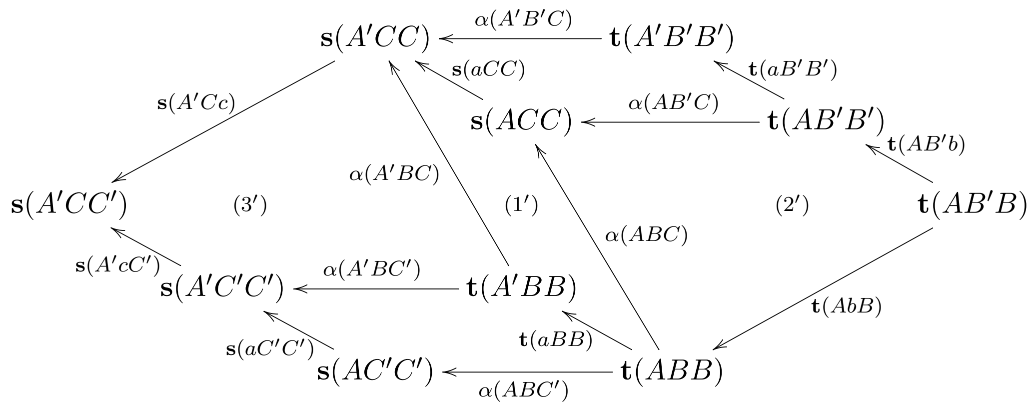


Figure 6: Making the Hexagon of Transformation

Perhaps it is worthwhile to go a little more carefully through one of those computations. Diagram (1) manifestly says that $\mathbf{E}\{[] \leftarrow \alpha(ABC)\} \circ \mathbf{s}([\,]CC) = \mathbf{E}\{\alpha(A'BC) \leftarrow []\} \circ \mathbf{t}([\,]BB)$; this means that $(\mathbf{E}\{[] \leftarrow \alpha(ABC)\} \circ \mathbf{s}([\,]CC))`a = (\mathbf{E}\{\alpha(A'BC) \leftarrow []\} \circ \mathbf{t}([\,]BB))`a$ for all $a\colon A' \longleftarrow A(\in \mathbf{A})$. Further, by definition of $\circ$ in $\mathbf{Set}$, we have $(g \circ f)`a = g`(f`a)$ so since $\mathbf{s}([\,]CC))`a = \mathbf{s}(aCC))$ by the action of the **hom** functor on arrows $\mathbf{E}\{[] \leftarrow \alpha(ABC)\}`\mathbf{s}(aCC) = [] \circ \mathbf{s}(aCC) \circ \alpha(ABC) = \mathbf{s}(aCC) \circ \alpha(ABC)$. Other equalities are similar.

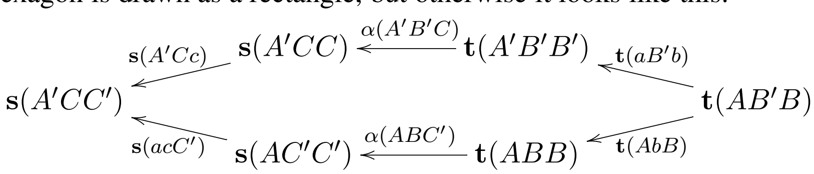
Diagram $(1')$ shows a natural transformation parameterized by B and C. Diagram $(2')$ shows a co-wedge transformation parameterized by A and C. Diagram $(3')$ shows a wedge transformation parameterized by A and B.

Note that the objects in the left and right of each diagram of the second triple (i.e. the target and source of the whole square) are target and source of the hom set on the left of the corresponding set diagram.

The advantage of the first three diagrams is that they do not involve the individual arrows, and therefore they also work in enriched categories (which need not have individual arrows).

Eilenberg and Kelly [14, (4),p368] display something which I call the "Hexagon of Transformation". They don't call it anything except "a single commuting diagram" into which (1), (2), (3) can be "condensed". I take the word "condensed" as a claim that the single diagram can be substituted for the original three to give an equivalent definition.

Their hexagon is drawn as a rectangle, but otherwise it looks like this:



(4)

If it is not obvious how the three diagrams (1'), (2'), (3') condense into the one hexagon, stare at Figure 6 and note that the three narrow parallelograms (forming a backward 'Z' shape in the middle) are instances of (1'), while the scalene quadrilaterals on the right and left are (2') and (3'), respectively.

In the other direction, the three diagrams are obtained from the Hexagon by setting
(1') $b = B' = B$ and $c = C' = C$, so $\mathbf{t}(ABB) = \mathbf{t}(AB'B)$ and $\mathbf{s}(A'CC) = \mathbf{s}(A'CC')$

(2') $a = A' = A$ and $c = C' = C$, so $\mathbf{s}(ACC) = \mathbf{s}(A'CC) = \mathbf{s}(AC'C) = \mathbf{s}(A'CC')$
(3') $a = A' = A$ and $b = B' = B$, so $\mathbf{t}(ABB) = \mathbf{t}(AB'B) = \mathbf{t}(A'B'B') = \mathbf{t}(A'B'B')$
respectively.

Given functors $\mathbf{s} : \mathbf{E} \leftarrow \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C}$ and $\mathbf{t} : \mathbf{E} \leftarrow \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B}$, using either the three squares or the hexagon, Eilenberg and Kelly define a transformation $\alpha : \mathbf{s} \Leftarrow \mathbf{t}$, to be an indexed family of arrows $\alpha(ABC) : \mathbf{s}(ACC) \leftarrow \mathbf{t}(ABB)$ that make the diagram(s) commute. (Actually, E&K does not call this a "transformation", but rather say that the family is "natural" in an extended sense.)

To return to Eilenberg and Kelly, E&K next defines (a) some special cases and (b) some further generalizations of the idea.

(a) For the special cases, let $\mathbf{I}$ be the category with one object and one arrow (the identity), both called $*$. If $\mathbf{B} = \mathbf{I}$ then $\mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B}$ can be identified with $\mathbf{A}$ and one can write $\mathbf{t}A = \mathbf{t}(A * *)$. If also $\mathbf{A} = \mathbf{I}$, write $S = \mathbf{t}(* * *)$. In a similar manner, setting two out of three at a time of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ to $\mathbf{I}$, we get definitions of extranatural transformations of three kinds:

$(1'')$ $\alpha_A : \mathbf{s}A \Leftarrow \mathbf{t}A$ when $\mathbf{s}, \mathbf{t} : \mathbf{E} \longleftarrow \mathbf{A}$ (natural transformation)

$(2'')$ $\alpha_B : S \Leftarrow \mathbf{t}(BB)$ when $\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{B}^{op} \times \mathbf{B}$ and $S \in \mathbf{E}$ (co-wedge transformation)

$(3'')$ $\alpha_C : \mathbf{s}(CC) \Leftarrow T$ when $\mathbf{s} : \mathbf{E} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$ and $T \in \mathbf{E}$ (wedge transformation)

The first of these is the ordinary definition of natural transformation. E&K cites Mac Lane's definition of "diagonal spread" [4, p54] as similar to the third definition, but a better reference now would be the definition of "wedge" CWM [24, §IX.4,p215]. We call it a "wedge transformation", because it is a special case of a general (extranatural) transformation. Case $(2'')$ has been called a "co-wedge" [22],although CWM calls them both "wedges".

(b) For the further generalization, pick three numbers, q, s, and t, and pretend that your numbers are substituted in the following to get a definition of a transformation as a family of $q+s+t$ arrows.

Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be the products $\mathbf{A} = \prod_{i<q} \mathbf{A}_i$, $\mathbf{B} = \prod_{j<t} \mathbf{B}_j$, and $\mathbf{C} = \prod_{k<s} \mathbf{C}_k$ in Definition 1 and let α be an indexed family of $q + s + t$ arrows:

$$\alpha(A_{..<q}, B_{..<t}, C_{..<s}) : \mathbf{s}(A_{..<q}, C_{..<s}, C_{..<s}) \longleftarrow \mathbf{t}(A_{..<q}, B_{..<t}, B_{..<t})$$

where $A_{..<q} = A_0, A_1, \ldots, A_{q-1}$; $B_{..<t} = B_0, B_1, \ldots, B_{t-1}$; $C_{..<s} = C_0, C_1, \ldots, C_{s-1}$:

$$\mathbf{s} : \mathbf{E} \longleftarrow \prod_{i<q} \mathbf{A}_i \times \prod_{k<s} \mathbf{C}_k^{op} \times \prod_{k<s} \mathbf{C}_k$$

and

$$\mathbf{t} : \mathbf{E} \longleftarrow \prod_{i<q} \mathbf{A}_i \times \prod_{j<t} \mathbf{B}_j^{op} \times \prod_{j<t} \mathbf{B}_j$$

so $\mathbf{s}$ is a functor of $q + 2s$ variables, and $\mathbf{t}$ is a functor of $q + 2t$ variables. E&K say that "the reader will easily verify" that α is a transformation in the sense of Definition 1 iff it is a transformation in each of its $q + t + s$ variables separately, in the sense of the special cases of (a) above, $(1'')$, $(2'')$, or $(3'')$, as appropriate, with the remaining variables held constant. To avoid disrupting the flow of the narrative, I have put this easy verification in claim 0.1 of section 2.4.2.

In the preceding I have changed the E&K notation $\mathbf{B}^*$ to $\mathbf{B}^{op}$, because it seems more clear. (We know what is meant by the opposite of a category.) I switch back for the next few pararaphs, because E&K use an asterisk for the dual and I am not clear what is meant by the dual. Kelly's book [19, p.12] holds some clues and subsection 2.4.2 attempts to clarify.

Still, E&K say "it is clear that" if $\mathbf{t}^* : \mathbf{E}^* \longleftarrow \mathbf{A}^* \times \mathbf{B} \times \mathbf{B}^*$ and $\mathbf{s}^* : \mathbf{E}^* \longleftarrow \mathbf{A}^* \times \mathbf{C} \times \mathbf{C}^*$ are the duals of $\mathbf{t}$ and $\mathbf{s}$ then $\alpha^* : \mathbf{t}^* \Leftarrow \mathbf{s}^*$ is a transformation. Note that α^* goes in reverse, though $\mathbf{t}^*$ and $\mathbf{s}^*$ do not.

"It is also clear that" if $\mathbf{p} : \mathbf{A} \longleftarrow \mathbf{A}'$, $\mathbf{q} : \mathbf{B} \longleftarrow \mathbf{B}'$, $\mathbf{r} : \mathbf{C} \longleftarrow \mathbf{C}'$, and $\mathbf{v} : \mathbf{E}' \longleftarrow \mathbf{E}$ are functors and α is a transformation then

$$\mathbf{v}\alpha(\mathbf{p}A', \mathbf{q}B', \mathbf{r}C') : \mathbf{vs}(\mathbf{p}A', \mathbf{r}^*C', \mathbf{r}C') \Leftarrow \mathbf{vt}(\mathbf{p}A', \mathbf{q}^*B', \mathbf{q}B')$$

is a transformation. This formula involves α, not α^*. Note that the primed letters are used sometimes for the source, sometimes for the target, as necessary to make the types of $\mathbf{t}$ and $\mathbf{s}$ stay as before. "Thus composition of transformations with functors poses no problems." All these non-problems are solved in subsection 2.4.2.

In contrast, a problem does arise when attempting to compose two transformations. The problem is that a transformation is a family of arrows indexed by arguments of the source and target functors. Attempted composition can result in loss of indices. For example, consider a functor $\mathbf{r} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{D}^{op} \times \mathbf{D}$ and a transformation $\beta(ACD) : \mathbf{r}(ADD) \Leftarrow \mathbf{s}(ACC)$. where $\mathbf{t}$, $\mathbf{s}$, and α are as in Definition 1, that is, $\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B}$, and $\mathbf{s} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C}$, and $\alpha(ABC) : \mathbf{s}(ACC) \Leftarrow \mathbf{t}(ABB)$..

Then the composition $\mathbf{r}(ADD) \xleftarrow{\beta(ACD)} \mathbf{s}(ACC) \xleftarrow{\alpha(ABC)} \mathbf{t}(ABB)$ may depend upon C, leaving no way to define $\gamma(ABD) = \beta(A?D) \circ \alpha(AB?) : \mathbf{r}(ADD) \Leftarrow \mathbf{t}(ABB)$.

However, if $\mathbf{C} = \mathbf{I}$, so that α depends only upon A and B and β upon A and D then the composition is a natural transformation as $\gamma(ABD) = \beta(AD) \circ \alpha(AB) : \mathbf{r}(ADD) \Leftarrow \mathbf{t}(ABB)$.

2.2 Sufficient conditions for composition

Returning to case (b) above, let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be the products $\mathbf{A} = \prod_{i<q} \mathbf{A}_i = \mathbf{A}_0 \times \cdots \times \mathbf{A}_{q-1}$, $\mathbf{B} = \prod_{j<t} \mathbf{B}_j = \mathbf{B}_0 \times \cdots \times \mathbf{B}_{t-1}$, and $\mathbf{C} = \prod_{k<s} \mathbf{C}_k = \mathbf{C}_0 \times \cdots \times \mathbf{C}_{s-1}$ and apply Definition 1 to obtain transformations $\alpha(A_{..<q}, B_{..<t}, C_{..<s}) : \mathbf{s}(A_{..<q}, C_{..<s}, C_{..<s}) \longleftarrow \mathbf{t}(A_{..<q}, B_{..<t}, B_{..<t})$ where $\mathbf{s}$ is a functor of $q + 2s$ arguments $\mathbf{s} : \mathbf{E} \longleftarrow \prod_{i<q} \mathbf{A}_i \times \prod_{k<s} \mathbf{C}_k^{op} \times \prod_{k<s} \mathbf{C}_k$ and $\mathbf{t}$ is a functor of $q + 2t$ arguments $\mathbf{t} : \mathbf{E} \longleftarrow \prod_{i<q} \mathbf{A}_i \times \prod_{j<t} \mathbf{B}_j^{op} \times \prod_{j<t} \mathbf{B}_j$ and α is a family of $q + s + t$ arrows of $\mathbf{E}$..

Now the $q + 2s$ arguments of $\mathbf{s}$ can be permuted and regrouped so that the arguments occur in pairs of opposite variance, thus: $\mathbf{s} : \mathbf{E} \longleftarrow \prod_{i<q} \mathbf{A}_i \times \prod_{k<s} (\mathbf{C}_k^{op} \times \mathbf{C}_k)$. Proper re-grouping will allow certain families of arrows, α, to be seen as transformations.

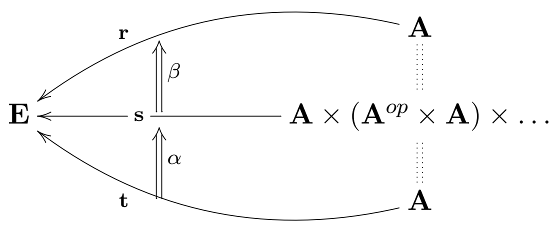
It might be instructive to skip the permutation and leave variables paired as $(C_{..<s}, C_{..<s}) = (C_0, C_{..\le s-1}, C_0, C_{..\le s-1})$. Does the notation work?

The general case reduces to three cases described in the three following propositions in which the composite $\beta \circ \alpha$ is defined. After the three cases we describe the general theory of Kelly-Mac Lane graphs, which gives more general sufficient conditions for a reduction to the three cases to work. The double dotted lines in the following three diagrams will turn out to be Kelly-Mac Lane graphs.

E&K say that it follows that these three cases are "essentially all the ones in which composition is defined." It is not clear what is meant by "essentially"; it certainly does not mean that the sufficient conditions are also necessary (see Remark 1 in Section 3.2).

Perversely, E&K call Proposition 3, Proposition 2*. It is the dual of Proposition 2, but the same is true of (3) with respect to (2), $(3')$ wrt $(2')$, and $(3'')$ wrt $(2'')$ above.

Vertical composition of ordinary natural transformations is as described in Proposition 1, with $n = 0$ so that all three functors $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$, are sourced from $\mathbf{A}$ so they are parallel.



($2n + 1$ arguments to $\mathbf{s}$)

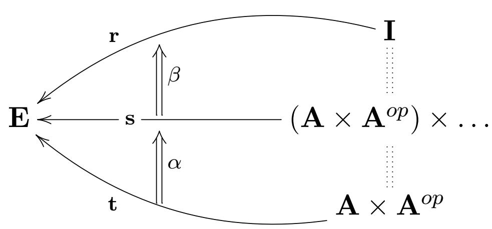
Proposition 1: *Given functors* $\mathbf{r} : \mathbf{E} \longleftarrow \mathbf{A}$ *and* $\mathbf{s} : \mathbf{E} \longleftarrow \mathbf{A} \times \prod_{i<n} (\mathbf{A}^{op} \times \mathbf{A})$ *and* $\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{A}$ *and transformations*

$$\alpha(AB_0 \dots B_{n-1}) : \mathbf{s}(AB_0B_0 \dots B_{n-1}B_{n-1}) \Leftarrow \mathbf{t}A$$

$$\beta(B_0 \dots B_{n-1}C) : \mathbf{r}C \Leftarrow \mathbf{s}(B_0B_0 \dots B_{n-1}B_{n-1}C)$$

the composition $\gamma(A) = \beta(A \dots A) \circ \alpha(A \dots A) : \mathbf{r}A \Leftarrow \mathbf{t}A$. *is an ordinary natural transformation.*

This is the same as saying that $\gamma(C) = \beta(C \dots C) \circ \alpha(C \dots C) : \mathbf{r}C \longleftarrow \mathbf{t}C$ is natural. Both A and C are variables.



($2n$ arguments to $\mathbf{s}$, $n > 0$)

Proposition 2: *Given functors* $\mathbf{r} : \mathbf{E} \longleftarrow \mathbf{I}$, $\mathbf{s} : \mathbf{E} \longleftarrow \prod_{i<n} (\mathbf{A} \times \mathbf{A}^{op})$, *and* $\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{A}^{op}$, *and transformations*

$$\alpha(AB_0 \dots B_{n-2}C) : \mathbf{s}(AB_0B_0 \dots B_{n-2}B_{n-2}C) \Leftarrow \mathbf{t}(AC)$$

$$\beta(B_0 \dots B_{n-1}) : \mathbf{r}(*) = R \Leftarrow \mathbf{s}(B_0B_0 \dots B_{n-1}B_{n-1}).$$

the composition $\gamma A = \beta(A \dots A) \circ \alpha(A \dots A) : R \Leftarrow \mathbf{t}(AA)$. *is a co-wedge transformation.*

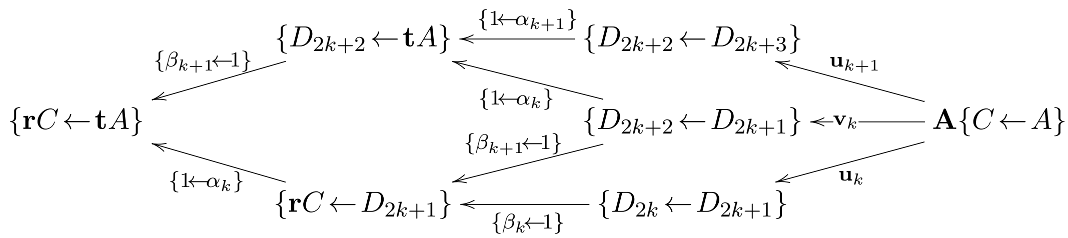
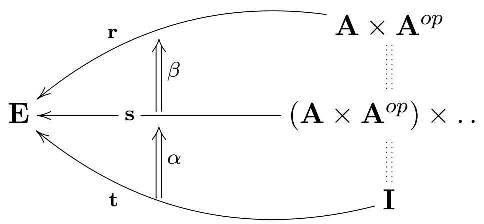


Figure 1: Inductive step $\{\beta_k \leftarrow \alpha_k\} \circ \mathbf{u}_k = \{\beta_{k+1} \leftarrow \alpha_k\} \circ \mathbf{v}_k = \{\beta_{k+1} \leftarrow \alpha_{k+1}\} \circ \mathbf{u}_{k+1}$



($2n$ arguments to $\mathbf{s}$, $n > 0$)

Proposition 3: *Given functors* $\mathbf{r} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{A}^{op}$, $\mathbf{s} : \mathbf{E} \longleftarrow \prod_{i<n} (\mathbf{A} \times \mathbf{A}^{op})$, *and* $\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{I}$, *and natural transformations*

$$\alpha(B_0 \dots B_{n-1}) : \mathbf{s}(B_0 B_0 \dots B_{n-1} B_{n-1}) \longleftarrow T = \mathbf{t}(*)$$

$$\beta(A B_0 \dots B_{n-2} C) : \mathbf{r}(AC) \longleftarrow \mathbf{s}(A B_0 B_0 \dots B_{n-2} B_{n-2} C)$$

the composition $\gamma A = \beta(A \dots A) \circ \alpha(A \dots A) : \mathbf{r}(AA) \Leftarrow T$ *is a wedge transformation,*

Proof. (of proposition 1): Refer to figure 1. Let's call this figure and its caption the inductive diagram and equation for Proposition 1. The proof will consist in proving the inductive equation, and then using induction on k (pasting together copies of the diagram) to prove it for all k, and then relating u_0 and u_n to $\mathbf{t}$ and to $\mathbf{r}$ to complete the proof.

An ordinary natural transformation is as described in Proposition 1,

Note that $A, B_k, C \in \mathbf{A}$. Define for $0 \le k \le n$ and $0 \le j \le 2n+1$

$D_j(AC) = \mathbf{s}(\underbrace{A \dots A}_{j} \underbrace{C \dots C}_{2n+1-j}) \in \mathbf{E}$

$\mathbf{u}_k(AC) = \mathbf{s}(\underbrace{A \dots A}_{2k} [\,] C \underbrace{C \dots C}_{2n-2k-1}) : \mathbf{E}\{D_{2k} \leftarrow D_{2k+1}\} \longleftarrow \mathbf{A}\{C \leftarrow A\}$

$\mathbf{v}_k(AC) = \mathbf{s}(\underbrace{A \dots A}_{2k} A [\,] \underbrace{C \dots C}_{2n-2k-1}) : \mathbf{E}\{D_{2k+2} \leftarrow D_{2k+1}\} \longleftarrow \mathbf{A}\{C \leftarrow A\}$ (where $k < n$)

$\alpha_k(AC) = \alpha(\underbrace{A \dots A}_{k} A \underbrace{C \dots C}_{n-k}) : D_{2k+1} \longleftarrow \mathbf{t}A$

$\alpha_-(AC) = \alpha(\underbrace{C \dots C}_{n+1}) : D_0 \longleftarrow \mathbf{t}C$

$\beta_k(AC) = \beta(\underbrace{A \dots A}_{k} C \underbrace{C \dots C}_{n-k)}) : \mathbf{r}C \longleftarrow D_{2k}$

$\beta_+(AC) = \beta(\underbrace{A \dots A}_{n+1}) : \mathbf{r}A \longleftarrow D_{2n+1}$

Those definitions are mostly from E&K, but D_k has been changed to D_j because the subscript needs to be greater than n sometimes. E&K do not write the indices (AC), and I will also leave them out most of the time, but it's important to remember that they are there. E&K do not explicitly define α_- and β_+ .

Note that all the αs and βs have $n+1$ indices each, and that α_k has $k+1$ As, while β_k has $n-k+1$ Cs. The arrow α_- might have been called α_{-1}, but the first index of α is also the argument of the source functor, so it is a bit of a special case. Similarly, β_+ might have been called β_{n+1}, but its last index is also the argument of the target functor. Furthermore, these indices are paired differently in the arguments to $\mathbf{s}$, which leads to different subscripts on D.

Now $\mathbf{u}_k$ is a covariant, and $\mathbf{v}_k$ a contravariant, functor, but Definition 1 (of transformation) applies to multi-argument (co/contra)-variant functors, so let's define one.

$\mathbf{w}_j = \mathbf{s}(\underbrace{A\ldots A}_{j}[\,][\,]\underbrace{C\ldots C}_{2n-j-1})$

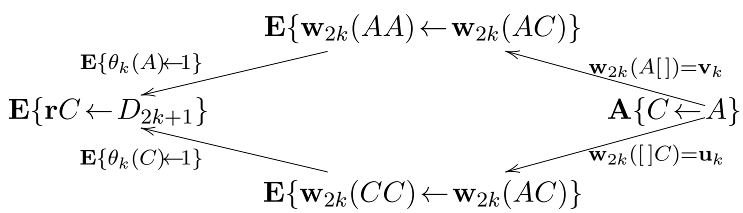
Now $\mathbf{w}_j$ is a bivalent functor, and if j is even it is covariant in the first argument and contravariant in the second. So $\mathbf{w}_{2k}(A[\,]) = \mathbf{v}_k$ and $\mathbf{w}_{2k}([\,]C) = \mathbf{u}_k$.

Also define $\theta_k = \beta(\underbrace{A\ldots A}_{k}[\,]\underbrace{C\ldots C}_{n-k-1}C)$ and $\phi_k = \alpha(A\underbrace{A\ldots A}_{k}[\,]\underbrace{C\ldots C}_{n-k-1})$,

so that $\theta_k(A) = \beta_{k+1}$ and $\theta_k(C) = \beta_k$ while $\phi_k(A) = \alpha_{k+1}$ and $\phi_k(C) = \alpha_k$.

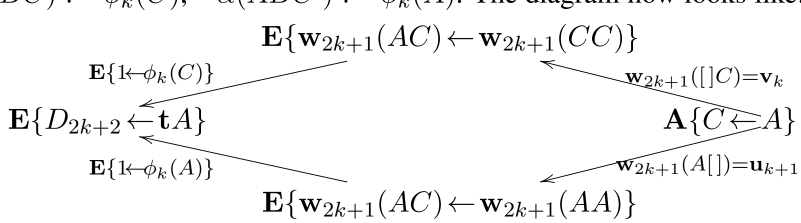
Now in Definition 1(2) (i.e. co-wedge) let $\mathbf{B} := \mathbf{A}$, $B' := C$, $B := A$, $\mathbf{t}(A[\,]B) := \mathbf{w}_{2k}(A[\,])$, $\mathbf{t}(AB'[\,]) := \mathbf{w}_{2k}([\,]C)$, $\alpha(ABC) := \theta_k(A)$, $\alpha(AB'C) := \theta_k(C)$. The variables on the left of the ":=" are those used in Definition 1, those on the right are those defined in the proof of Proposition 1 (i.e. the few paragraphs above).

After those substitutions, diagram (2) of Definition 1 looks like:



Note that $\mathbf{w}_{2k}(CC) = D_{2k}$ and $\mathbf{w}_{2k}(AC) = D_{2k+1}$ and $\mathbf{w}_{2k}(AA) = D_{2k+2}$ This shows that the bottom "square" on the right of the diagram of Figure 1 fits the definition of a co-wedge transformation.

Next, in definition 1(3) (i.e. wedge transformation) let $\mathbf{C} := \mathbf{A}$, $C' := C$, $C := A$, $\mathbf{s}(A[\,]C') := \mathbf{w}_{2k+1}([\,]C)$, $\mathbf{s}(AC[\,]) := \mathbf{w}_{2k+1}(A[\,])$, $\alpha(ABC) := \phi_k(C)$, $\alpha(ABC') := \phi_k(A)$. The diagram now looks like:

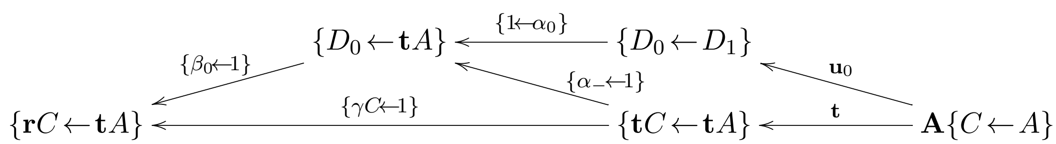


Note that $\mathbf{w}_{2k+1}(CC) = D_{2k+1}$ and $\mathbf{w}_{2k+1}(AC) = D_{2k+2}$ and $\mathbf{w}_{2k+1}(AA) = D_{2k+3}$. This shows that the square on the top right of Figure 1 fits the definition of a wedge transformation. The left square of Figure 1 commutes because

$$\{1 \leftarrow \alpha_k\} \circ \{\beta_{k+1} \leftarrow 1\} = \{\beta_{k+1} \leftarrow \alpha_k\} = \{\beta_{k+1} \leftarrow 1\} \circ \{1 \leftarrow \alpha_k\}$$

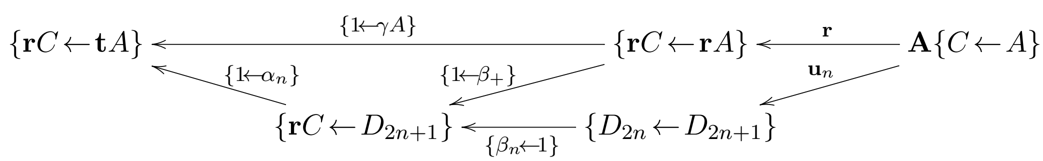
so the whole diagram commutes, which shows that the equation under it is correct. By induction and transitivity of equality, conclude that for all $k \leq n$. $\{\beta_k \leftarrow \alpha_k\} \circ \mathbf{u}_k = \{\beta_0 \leftarrow \alpha_0\} \circ \mathbf{u}_0$.

Now the following diagram commutes because, by hypothesis, α is a natural transformation to $\mathbf{s}([\,]C \ldots C) = \mathbf{u}_0$ from $\mathbf{t}$, while $\gamma(C) = \beta(C \ldots C)\alpha(C \ldots C) = \beta_0 \circ \alpha_-$:



Since the diagram commutes, $\{\gamma C \leftarrow 1\} \circ \mathbf{t} = \{\beta_0 \leftarrow \alpha_0\} \circ \mathbf{u}_0$.

For similar reasons:



Since this diagram commutes, $\{1 \leftarrow \gamma A\} \circ \mathbf{r} = \{\beta_n \leftarrow \alpha_n\} \circ \mathbf{u}_n$.

Finally, putting it all together, we have the (ordinary) naturality condition for γ,

$$\mathbf{E}\{\gamma(C) \leftarrow 1\} \circ \mathbf{t} = \mathbf{E}\{1 \leftarrow \gamma(A)\} \circ \mathbf{r} \ .$$ □

Proof. (of proposition 2):

Refer to Figure 2. Define D_j, $\mathbf{u}_k$, and $\mathbf{v}_k$ the same as in the previous proof, except that there are fewer arguments to $\mathbf{s}$, and therefore one fewer "C" to pad out the end. So define for $0 \leq k < n$ and $0 \leq j \leq 2n$

$$D_j = \mathbf{s}(\underbrace{A \ldots A}_{j}\underbrace{C \ldots C}_{2n-j}) \in \mathbf{E}$$

$$\mathbf{u}_k = \mathbf{s}(\underbrace{A \ldots A}_{2k}[\,]\underbrace{C \ldots C}_{2n-2k-1}) : \mathbf{E}\{D_{2k} \leftarrow D_{2k+1}\} \longleftarrow \mathbf{A}\{C \leftarrow A\}$$

$$\mathbf{v}_k = \mathbf{s}(\underbrace{A \ldots A}_{2k+1}[\,]\underbrace{C \ldots C}_{2n-2k-2}) : \mathbf{E}\{D_{2k+2} \leftarrow D_{2k+1}\} \longleftarrow \mathbf{A}\{C \leftarrow A\} \quad \text{(where } k < n\text{)}$$

The natural transforms are a little different. The target and source differ, and β has one fewer index because it no longer depends upon C.

$$\alpha_k = \alpha(A\underbrace{A \ldots A}_{k}\underbrace{C \ldots C}_{n-k}) : D_{2k+1} \longleftarrow \mathbf{t}AC$$

$$\alpha_- = \alpha(\underbrace{C \ldots C}_{n+1}) : D_0 \longleftarrow \mathbf{t}AC$$

$$\beta_k = \beta(\underbrace{A \ldots A}_{k}\underbrace{C \ldots C}_{n-k-1)}C) : R \longleftarrow D_{2k}$$

$$\beta_+ = \beta(\underbrace{A \ldots A}_{n}) : R \longleftarrow D_{2n+1}$$

With these changes to the definitions, the inductive diagram and equation are similar, and the proof is the same that for all $k < n$. $\{\beta_k \leftarrow \alpha_k\} \circ \mathbf{u}_k = \{\beta_0 \leftarrow \alpha_0\} \circ \mathbf{u}_0$.

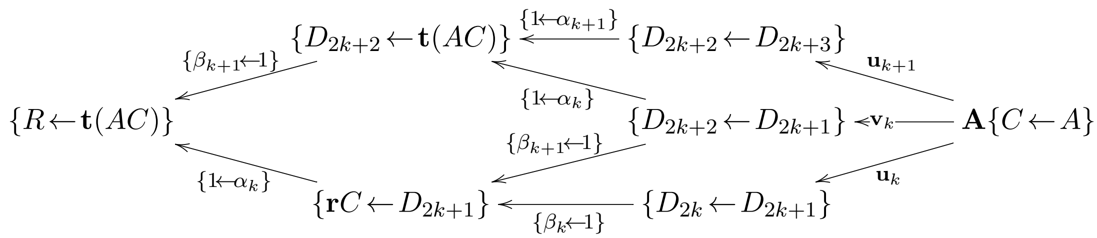


Figure 2: Inductive step $\{\beta_k \leftarrow \alpha_k\} \circ \mathbf{u}_k = \{\beta_{k+1} \leftarrow \alpha_k\} \circ \mathbf{v}_k = \{\beta_{k+1} \leftarrow \alpha_{k+1}\} \circ \mathbf{u}_{k+1}$

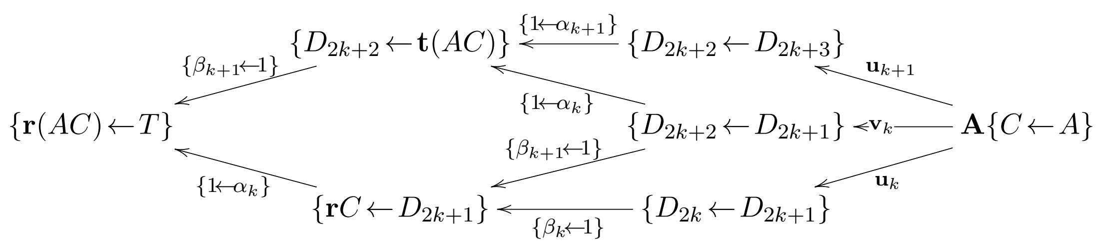
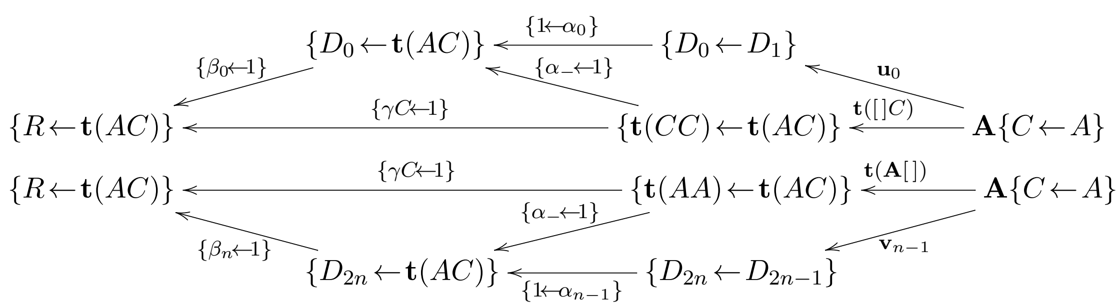


Figure 3: Inductive step $\{\beta_k \leftarrow \alpha_k\} \circ \mathbf{u}_k = \{\beta_{k+1} \leftarrow \alpha_k\} \circ \mathbf{v}_k = \{\beta_{k+1} \leftarrow \alpha_{k+1}\} \circ \mathbf{u}_{k+1}$



And so $\{\gamma C \leftarrow 1\} \circ \mathbf{t}(A[]) = \{\gamma C \leftarrow 1\} \circ \mathbf{t}([]C)$

□

Proof. (of proposition 3): See figure 3. This is the dual of the previous proposition, and its proof is the dual of the previous proof. □

2.3 Kelly-Mac Lane graphs

At this point E&K introduce Kelly-Mac Lane graphs, although, of course, they do not use that name. Although Eilenberg and Kelly first described these graphs, they have come to be called Kelly-Mac Lane graphs because their paper (K&M[20]) used them to prove an important theorem, *to wit* coherence for symmetric monoidal closed categories.

Let X be a finite set with members called vertices. A graph on X is a set of edges, each of which is incident on a pair of distinct vertices $x, y \in X$. Each of the two vertices are also said to be incident on the edge. So these are undirected graphs, that is, an an edge is incident on vertices x and y iff it is incident on y and x. Parallel edges are allowed, but one edge loops are not. A graph is called a simple graph, or a matching if each vertex is incident on exactly one edge. The two vertices incident on one edge are called mates. (Each is the mate of the other.)[5] There is a matching on X iff the cardinality of X is even.

Now define a category which E&K call $\mathbf{X}$. An object of $\mathbf{X}$ is a finite set. If $X, Y \in \mathbf{X}$, let $Y + X$ denote the disjoint union of Y and X. (If $\mathbf{X}$ seems too large, you may get by with its skeleton, which is the small category of finite ordinals, with $+$ meaning addition. I continue with E&K approach.) An arrow $f : Y \longleftarrow X$ in $\mathbf{X}$ is a matching on $Y + X$. The identity $X : X \longleftarrow X$ has X itself as edges. Each edge is incident on the two copies of itself in $X + X$. If $g : Z \longleftarrow Y$ then $g \circ f$ is the matching on $Z + X$ "induced" by $g + f$.

Here $g + f$ is the disjoint union of the edges of g and f on $Z + Y + X$ with the same incidence relation, where there is only one copy of Y. This is no longer a matching, but it is a graph with no vertex incident to more than two or fewer than one edge. Indeed, every vertex in Y is incident on exactly two edges, one from f and one from g, while each vertex in $Z + X$ is incident on exactly one edge. So the graph consists of paths, each of which is a sequence of edges alternately from g and f, with each successive pair of edges incident on a single vertex. A path is either an arc or a closed loop. If a vertex is incident on just one edge then it is an endpoint of an arc, which has one other endpoint. The endpoints of an arc, if any, are in Z or X. A one edge arc, where the only vertices are endpoints, can be entirely in X or in Z, but if there are intermediate vertices, they must be in Y.

The matching induced by $g + f$ is determined by its connected components, which are paths of the following four kinds:

1. an arc with one endpoint in X and the other in Z
2. an arc with both endpoints in X
3. an arc with both endpoints in Z
4. a closed loop entirely in Y.

In kinds 2 and 3, the arc may have only one edge, but there are an odd number of edges, and any inermediate edges have both endpoints in Y.

Then $g \circ f$, the matching induced by $g + f$, is the matching on $Z + X$ which results by ignoring closed loops and putting an edge between the two endpoints of any arc (path of the first three kinds).

To show that this defines a category, the identity and associative laws must be verified. The identity matching on X was defined above as $X + X$ with each member of X matched with itself in the other component of the disjoint union. It is easy to see that that is an identity of the composition just defined. Suppose, to verify the associative law that $g : Z \longleftarrow Y$, $f : Y \longleftarrow X$, and $h : X \longleftarrow W$. The graph $g + f + h$ on $Z + Y + X + W$ also consists of arcs and loops, ignoring the loops and replacing arcs with edges gives $g \circ f \circ h : X \longleftarrow W$. Each of those arcs of $g + f + h$ consists of arcs of g alternating with arcs of $f + h$, and so $g \circ f \circ h = g \circ (f \circ h)$ and similarly, $g \circ f \circ h = (g \circ f) \circ h$.

Call the arrows $g : Z \longleftarrow Y$ and $f : Y \longleftarrow X$ compatible if $g + f$ has no loops. Further, assuming $h : X \longleftarrow W$, then g, f, h are compatible if $g + f + h$ has no loops. The compatible sequences

[5] E&K do not use the terms "mate" and "matching'; they say "simple graph". Mates are mentioned in K&M[20].

of arrows do *not* form a category, because it is possible for g, f and f, h to be compatible, but g, f, h to be not compatible. Nevertheless, the following three conditions are equivalent.

1. g, f, h are compatible
2. g, f are compatible and $g \circ f, h$ are compatible
3. f, h are compatible and $g, f \circ h$ are compatible

It will turn out that compatible graphs correspond to extranatural transformations.

Now define a large category which E&K call $\mathbf{Y}$. An object in $\mathbf{Y}$ is family of categories indexed by an object of $\mathbf{X}$ (e.g. if $X \in \mathbf{X}$, so that X is a finite set, and for each $x \in X$, $\mathbf{B}_x$ is a category, then $B_X = \{\mathbf{B}_x | x \in X \in \mathbf{X}\}$ is an object $B_X \in \mathbf{Y}$).

(This whole construction could be generalized by using only a subcategory of the category of categories to construct the category composed of Matchings of Finite Families of (some specified) categories. Then $\mathbf{Y} = \mathbf{MFF}(\mathbf{Cat})$). I continue with the E&K approach.)

If $C_Y = \{\mathbf{C}_y | y \in Y \in \mathbf{X}\}$ is also an object $C_Y \in \mathbf{Y}$ then an arrow $f : C_Y \longleftarrow B_X$ is a triple $f = (C_Y, m, B_X)$ where $m : Y \longleftarrow X$ is a matching in $\mathbf{X}$ such that every pair of mates is one of the following three types

1. if $x \in X, y \in Y$ are mates then $\mathbf{B}_x = \mathbf{C}_y$
2. if $x, x' \in X$ are mates then $\mathbf{B}_x = \mathbf{B}_{x'}^{op}$
3. if $y, y' \in Y$ are mates then $\mathbf{C}_y = \mathbf{C}_{y'}^{op}$

Composition is defined by $(D_Z, n, C_Y) \circ (C_Y, m, B_X) = (D_Z, n \circ m, B_X)$ where the composition of matchings on the right side is in $\mathbf{X}$.

The forgetful functor $\mathbf{u} : \mathbf{X} \longleftarrow \mathbf{Y}$ defined by $\mathbf{u}`B_X = X$ and $\mathbf{u}(B_X, m, C_Y) = m$ is faithful.

There seem to be several misprints in this penultimate page of E&K. For example q is used sometimes as an index variable, sometimes as the upper bound on the index; the varable r seems to be used exactly once, without definition. It is not possible for me to list these misprints, or even be sure they are real, because of the poor quality of my copy. After smudging, microfilming, enlarging, digitizing and printing, some of the subscripts are just fuzzy grey blobs. I will write what I think it should say.

A $\mathbf{EY}$-functor[6] $\mathbf{t} : \mathbf{E} \longleftarrow B_X$, where $B_X \in \mathbf{Y}$, is defined to be a functor $\mathbf{t} : \mathbf{E} \longleftarrow \prod_{x \in X} \mathbf{B}_x$.

If (C_Y, m, B_X) is an arrow in $\mathbf{Y}$ and $\mathbf{t} : \mathbf{E} \longleftarrow B_X$ and $\mathbf{s} : \mathbf{E} \longleftarrow C_Y$ are $\mathbf{EY}$-functors, the variables can be renamed and reordered so that there are ordinary functors, which are denoted by the same letters as the corresponding $\mathbf{EY}$-functors:

$$\mathbf{t} : \mathbf{E} \longleftarrow \prod_{i<q} \mathbf{A}_i \times \prod_{j<t} (\mathbf{B}_j^{op} \times \mathbf{B}_j)$$

$$\mathbf{s} : \mathbf{E} \longleftarrow \prod_{i<q} \mathbf{A}_i \times \prod_{k<s} (\mathbf{C}_k^{op} \times \mathbf{C}_k)$$

[6] E&K call this simply a functor, but it's not, is it?

where the $\mathbf{A}_i$ correspond to mates of type 1 ($\mathbf{A}_i = \mathbf{B}_x = \mathbf{C}_y$ for some x and y), the $\mathbf{B}_j^{op}$, $\mathbf{B}_j$ pairs correspond to mates of type 2 ($\mathbf{B}_j = \mathbf{B}_x$ and $\mathbf{B}_j^{op} = \mathbf{B}_{x'}$), and the $\mathbf{C}_k^{op}$, $\mathbf{C}_k$ pairs correspond to mates of type 3 ($\mathbf{C}_k = \mathbf{C}_y$ and $\mathbf{C}_k^{op} = \mathbf{C}_{y'}$), in the matching m which is part of the arrow (C_Y, m, B_X). We have q mates of type (1), s mates of type (2), and t mates of type (3). Now Propositions 1, 2, and 3 above apply to show that there is a transformation $\alpha :: \mathbf{s} \Longleftarrow \mathbf{t}$, which is called a transformation of type m.

The components of $\alpha :: \mathbf{s} \Longleftarrow \mathbf{t}$ are

$$\alpha(A_{..<q}, B_{..<t}, C_{..<s}) : \mathbf{s}(A_{..<q}, (CC)_{..<s}) \longleftarrow \mathbf{t}(A_{..<q}, (BB)_{..<t})$$

where by $(BB)_{..<t}$ is meant $B_0, B_0, B_1, B_1, \ldots, B_{t-1}, B_{t-1}$, each variable occurs twice, once in a covariant, once in a contravariant, position.

Now consider arrows (D_Z, n, C_Y) and (C_Y, m, B_X) in $\mathbf{Y}$, and $\mathbf{EY}$-functors

$$\mathbf{r} : \mathbf{E} \longleftarrow D_Z \quad \mathbf{s} : \mathbf{E} \longleftarrow C_Y \quad \mathbf{t} : \mathbf{E} \longleftarrow B_X$$

and natural transformation $\beta :: \mathbf{r} \Longleftarrow \mathbf{s}$ and $\alpha :: \mathbf{s} \Longleftarrow \mathbf{t}$ of types m and n, respectively.

Assuming the graph $m + n$ is connected, if it consists of a single arc, then we are in a situation in which one of the three propositions on composition apply, and so $\beta \circ \alpha :: \mathbf{r} \Longleftarrow \mathbf{t}$ is a natural transformation of type $m \circ n$.

Still assuming the graph $m + n$ is connected, if it consists of a closed curve, then we are in a situation like

$$\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{I} \quad \mathbf{s} : \mathbf{E} \longleftarrow \prod_{i<s} (\mathbf{A} \times \mathbf{A}^{op}) \quad \mathbf{r} : \mathbf{E} \longleftarrow \mathbf{I}$$

$$\alpha(A_0 \ldots A_{s-1}) :: \mathbf{s}(A_0 A_0 \ldots A_{s-1} A_{s-1}) \Longleftarrow \mathbf{t}$$

$$\beta(A_0 \ldots A_{s-1}) :: \mathbf{r} \Longleftarrow \mathbf{s}(A_0 \ldots A_{s-1} A_{s-1} A_0)$$

and the composition $\beta \circ \alpha :: \mathbf{r} \Longleftarrow \mathbf{t}$ may depend upon the A_i, and so will not in general be natural. Still, as K&M point out (see section 3.2, Remark 1) there may be further special cases in which the compostion can be defined so as to be natural.

If the graph $m + n$ is not connected then the preceding analysis can be applied to each of the connected components separately, setting constant the variables corresponding to the other components. So if m and n are compatible, then the compostion $\beta \circ \alpha$ is a transformation in sense defined above, that is, it is a combination of ordinary natural transformation and dinatural transformations.

E&K ends by saying that a followup paper [1] describes natural transformations in the setting of enriched category theory[7]. In this generalized setting, the diagrams (1), (2), and (3) of definition 1 can be used nearly unchanged, while the diagrams (1'), (2'), and (3') make no sense. Kelly's book [19, §1.7] also describes extraordinary natural transformations in enriched category theory.

2.4 Questions? Comments?

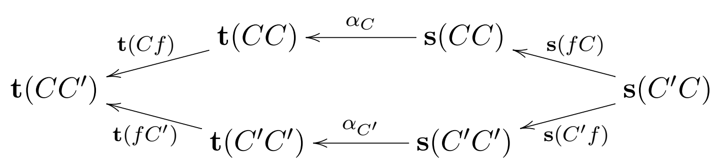
In this section I quit following E&K. These are my own questions and comments.

[7]This paper, *Closed Categories* [13] is described in section 4.3, page 31 of these notes.

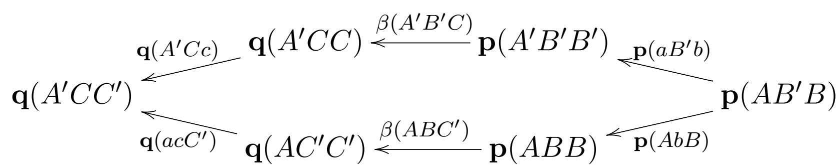
2.4.1 Dinatural and Extranatural Transformations

Saunder Mac Lane, in his famous book *Categories for the Working Mathematician*, defines a dinatural transformation. Diagonal naturality as described by Mac Lane [24, CWM §IX.4] is a special case of extranaturality as defined by Eilenberg and Kelly [14].
NB: CWM uses T for the target and S for the source of the dinatural transformation. This reasonable convention is the opposite of that originally used by Eilenberg and Kelly [14].

Given parallel functors $\mathbf{s}, \mathbf{t} : \mathbf{E} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$, a dinatural transformation $\alpha : \mathbf{t} \overset{\bullet\bullet}{\longleftarrow} \mathbf{s}$ in the sense of Mac Lane [24, CWM §IX.4] is a family of arrows $\alpha_C : \mathbf{t}(C, C) \longleftarrow \mathbf{s}(C, C)$ indexed by objects $C \in \mathbf{C}$ such that for every arrow $f : C' \longleftarrow C (\in \mathbf{C})$ the hexagon of dinaturality commutes. Mac Lane's hexagon of dinaturality looks like this:



To avoid confusion, let's rename the functors in the E&K definition of a transformation[8] so that $\beta : \mathbf{q} \Leftarrow \mathbf{p}$ where $\mathbf{q} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C}$ and $\mathbf{p} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B}$:



Let $\mathbf{I}$ be the category with one object and one arrow (the identity), both called $*$.

The following theorem can be summarized as saying that a dinatural transformation is a special case of (extranaural) transformation. Specifically, it is a transformation where the categories $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, have a special form, and the indexing of the transformational family of arrows is also specialized in a way that is justified by the special form of those three categories.

Theorem 1. *Suppose we are given functors* $\mathbf{q} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C}$*, and* $\mathbf{p} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B}$ *where* $\mathbf{B} = \mathbf{C}^{op}$ *and* $\mathbf{A} = \mathbf{I}$*. If* $\beta : \mathbf{q} \Leftarrow \mathbf{p}$ *is a transformation then the family* $\alpha_C = \beta(*CC)$ *is dinatural* $\alpha : \mathbf{s} \overset{\bullet\bullet}{\longleftarrow} \mathbf{t}$ *where* $\mathbf{s}, \mathbf{t} : \mathbf{E} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$*, and* $\mathbf{t}(xy) = \mathbf{q}(*xy)$*, and* $\mathbf{s}(yx) = \mathbf{p}(*xy)$*.*

Proof. Assume given $\mathbf{q}$ and $\mathbf{p}$ as above and that $\beta : \mathbf{q} \Leftarrow \mathbf{p}$ is a transformation. Define $\mathbf{s}$, $\mathbf{t}$, and α as above; Note that the arguments of $\mathbf{s}$ are swapped while those of $\mathbf{t}$ are not.

It must be shown that the dinaturality hexagon commutes for every $f : C' \longleftarrow C$. For any such f, let $b = f$ and $c = f$. Since $\mathbf{B} = \mathbf{C}^{op}$, $\mathbf{B}$ and $\mathbf{C}$ have the same objects, so set $B = C'$ and $B' = C$, so that, $b = f : C \longleftarrow C' (\in \mathbf{C}^{op})$ or equivalently $b : B' \longleftarrow B (\in \mathbf{B})$ and $c : C' \longleftarrow C (\in \mathbf{C})$.

Now $\mathbf{q} : \mathbf{E} \longleftarrow \mathbf{I} \times \mathbf{C}^{op} \times \mathbf{C}$ and $\mathbf{p} : \mathbf{E} \longleftarrow \mathbf{I} \times \mathbf{B}^{op} \times \mathbf{B}$ equivalently $\mathbf{p} : \mathbf{E} \longleftarrow \mathbf{I} \times \mathbf{C} \times \mathbf{C}^{op}$ and so $\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$ and $\mathbf{s} : \mathbf{E} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$.

[8] By happy coincidence the object names exactly match those of Loregian's diagram (1.16) [22, p.7].

Now we must show that the dinaturality hexagon commutes, that is

$$\begin{array}{lll} \mathbf{t}(Cf) \circ \alpha_C \circ \mathbf{s}(fC) = \mathbf{t}(fC') \circ \alpha_{C'} \circ \mathbf{s}(C'f) & & \text{which is} \\ \mathbf{q}(*Cf) \circ \beta(*CC) \circ \mathbf{p}(*Cf) = \mathbf{q}(*fC') \circ \beta(*C'C') \circ \mathbf{p}(*fC') & & \text{which is} \\ \mathbf{q}(*Cc) \circ \beta(*B'C) \circ \mathbf{p}(*B'b) = \mathbf{q}(*cC') \circ \beta(*BC') \circ \mathbf{p}(*bB) & & \end{array}$$

which is exactly the hexagon of transformation. □

Well, maybe not exactly. According to CWM [24, §II.1.2] the category $\mathbf{C}^{op}$ has arrows f^{op}. I wanted to write $b = f^{op}$ above but have not yet made that work. Maybe I'm not smart enough, but maybe Mac Lane's hexagon does not follow Mac Lane's convention. In particular, "f" occurs four times, twice where an arrow of $\mathbf{C}^{op}$ is expected and twice where an arrow of $\mathbf{C}$ is expected. Is $f = f^{op}$ after all, or is there an application of a bijection which is not explicitly written?

2.4.2 Clear and Easily Seen

In this subsection we go back to fill in some details of things that E&K leave sketchy. Most of them remain sketchy. Actually, they are mostly placeholders for things that should be sketched better.

As noted on page 9 above, E&K say that "the reader will easily verify" the following claim:

Claim 0.1. *If α is an indexed family of arrows*

$$\alpha(A_{..<q}, B_{..<t}, C_{..<s})\colon \mathbf{s}(A_{..<q}, C_{..<s}, C_{..<s}) \longleftarrow \mathbf{t}(A_{..<q}, B_{..<t}, B_{..<t})$$

where

$$\mathbf{t}\colon \mathbf{E} \longleftarrow \prod_{i<q} \mathbf{A}_i \times \prod_{j<t} \mathbf{B}_j^{op} \times \prod_{j<t} \mathbf{B}_j$$

and

$$\mathbf{s}\colon \mathbf{E} \longleftarrow \prod_{i<q} \mathbf{A}_i \times \prod_{k<s} \mathbf{C}_k^{op} \times \prod_{k<s} \mathbf{C}_k$$

then α is natural in the sense of Definition 1 iff it is natural in each of its $q + t + s$ variables separately (in the sense of $(1'')$, $(2'')$, or $(3'')$, as appropriate) with the remaining variables held constant.

Also "it is clear that"

Claim 0.2. *With α, $\mathbf{t}$, and $\mathbf{s}$ as above, and α natural, if*

$$\mathbf{t}^*\colon \mathbf{E}^* \longleftarrow \mathbf{A}^* \times \mathbf{B} \times \mathbf{B}^*$$

and

$$\mathbf{s}^*\colon \mathbf{E}^* \longleftarrow \mathbf{A}^* \times \mathbf{C} \times \mathbf{C}^*$$

are the duals of $\mathbf{t}$ and $\mathbf{s}$ then $\alpha^* :: \mathbf{t}^* \Longleftarrow \mathbf{s}^*$ is natural.

Note that α^* goes in reverse, though $\mathbf{t}^*$ and $\mathbf{s}^*$ do not.

"It is also clear that"

Claim 0.3. *If* $\mathbf{p} : \mathbf{A} \longleftarrow \mathbf{A}'$, $\mathbf{q} : \mathbf{B} \longleftarrow \mathbf{B}'$, $\mathbf{r} : \mathbf{C} \longleftarrow \mathbf{C}'$, *and* $\mathbf{v} : \mathbf{E}' \longleftarrow \mathbf{E}$ *then* $\mathbf{v} \circ \alpha(\mathbf{p}A', \mathbf{q}B', \mathbf{r}C') :: \mathbf{v} \circ \mathbf{s}(\mathbf{p}A', \mathbf{r}^*C', \mathbf{r}C') \Longleftarrow \mathbf{v} \circ \mathbf{t}(\mathbf{p}A', \mathbf{q}^*B', \mathbf{q}B') : \mathbf{E}' \longleftarrow \mathbf{A}' \times \mathbf{B}' \times \mathbf{C}'$ *is natural (in* A'*, and* B'*, and* C'*).*

What does “dual” mean? The dual of a category is the opposite category, but what exactly is the dual of a functor or a natural transformation?

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