Notes on *CoEnd Calculus* *

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必也正名乎 — 孔子
Most imperative is to rectify names.
— Confucius [5, XIII (Zi Lu) §3]

There is only one thing worse than bad terminology,
and that is *continually changing* terminology.
— van Oosten [20, p.vii]

1 Newer Notes

These are my notes on *Coend calculus*, by Fosco Loregian [17], together with old notes which have been revised in light of what I learned from, or figured out while, reading that book. I have read only a small part of the book, and understood only a small part of that. Since *Coend Calculus* is still under construction, there may be an advantage to making these notes available. I have found a few local misprints that could be easily fixed before the text is fixed[1]. I intend to put this document on the web server, and update it from time to time, but make no promises.

The first section is a quick overview of some of the notational conventions I use followed by comments on specific pages of of the book, and then some longer and more general comments.

The second section consists of extracts from old notes updated. Since those notes are on decades-old papers, there was never any urgency to make them available, but there will probably eventualy be some of them that I put in the same place.

Most of this is either well-known or wrong. All of it could be read as a case study of the kind of confusion and misunderstanding that can arise. That may be useful to one trying to teach the subject. If there is anything new, perhaps it is in some of the diagrams. Look at the pictures on pages 22, 31, and 37.

This was orignally written with reference to Version 5 (v5) of the book. When I discovered there is now (v6), I updated page numbers, no doubt I missed some.

Hope this helps; use at your own risk!

*Draft, revision 5, please do not copy.
Instead fetch the latest version from `www.free-comp-shop.com/notes.html`

[1] ... repaired before finalized. There. I fixed the footnote.

Contents

1.1 Notation and Terminology

The apostrophe in "$R{‘}y$" may be read "of".
— Whitehead and Russell[25, *30]

The first notational problem anyone writing about category theory faces is whether composition should be written left-to-right or right-to-left. On the one hand, right-to-left composition has a long history in all branches of mathematics, but left-to-right composition seems more consistent with the diagrams. I want something that works for everyone, and I don't want to be the one to tell all the sophmores and engineers in the world, that, despite what all the books say, from now on we write ${}^{ix}e = (x)\cos + (x)\sin i$. We will stay with the long tradition of right-to-left composition, but make the formulas match the diagrams by reversing the diagrams, The arrows in a diagram point left whenever possible, as shown in Figures 1 and 2.

Following Whitehead and Russell [25], an apostrophe symbolizes the application of a function (or functor) e.g. $\cos{‘}\pi = -1$. A small circle symbolizes composition. Thus: $(f \circ g){‘}x = f{‘}(g{‘}x)$. Either of those might be omitted, but I try to avoid omitting two different symbols in the same formula. Sometimes the apostrophe is omitted and redundant parenthesis are added: $\cos(\pi)$. Also following Whitehead and Russell $f{“}S = \{f{‘}x | x \in S\}$. Following Barendregt [2, §2.1.18], sometimes $[\,]$ is a hole into which a parameter can be placed.

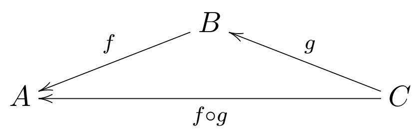


Figure 1: Composition

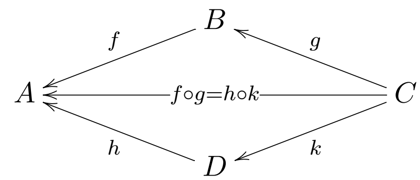


Figure 2: Equality

In addition, when discussing abstract categories, letters are: $\star$ Upper case for categories and objects (or identities); $\star$ Lower case for functors and arrows; $\bullet$ Bold for categories and functors; $\bullet$ Light for objects and arrows; $\bullet$ Greek for (natural) transformations; Also $[]$ is an identity arrow, 1 is a terminal object.

The "hom" functor for a category $\mathbf{C}$ is $\mathbf{C}\{[\,]\leftarrow[\,]\}\,\mathbf{:}\,\mathbf{Set} \longleftarrow \mathbf{C}\times\mathbf{C}^{op}$. The name of the category $\mathbf{C}$ can be omitted if it is understood. If $A, B \in \mathbf{C}$, then $\mathbf{C}\{A\leftarrow B\} = \{f | f\,\mathbf{:}\, A \longleftarrow B \in \mathbf{C}\}$. As a functor, hom also applies to arrows: given two arrows $a\,\mathbf{:}\,A \longleftarrow A'$ and $b\,\mathbf{:}\,B' \longleftarrow B$, we define $\{a\leftarrow b\}\,\mathbf{:}\,\{A\leftarrow B\} \longleftarrow \{A'\leftarrow B'\}$ by $\{a\leftarrow b\}\text{‘}f = a\circ f\circ b \in \{A\leftarrow B\}$ for every $f \in \{A'\leftarrow B'\}$.

Sometimes the identity arrow on an object has the same name as the object.

If $A\,\mathbf{:}\,A \longleftarrow A$ and $B\,\mathbf{:}\,B \longleftarrow B$, then $\{A\leftarrow B\}$ maps any $f \in \{A\leftarrow B\}$ to $A\circ f\circ B = f \in \{A\leftarrow B\}$. Thus, $\{A\leftarrow B\}$ is the identity function on the set of arrows of the same name.

For $a\,\mathbf{:}\,A \longleftarrow A'$, $\quad b\,\mathbf{:}\,B' \longleftarrow B$, and $\quad a'\,\mathbf{:}\,A' \longleftarrow A''$, $\quad b'\,\mathbf{:}\,B'' \longleftarrow B'$, we have

$$\{a\leftarrow b\}\,\mathbf{:}\,\{A\leftarrow B\} \longleftarrow \{A'\leftarrow B'\}, \qquad \{a'\leftarrow b'\}\,\mathbf{:}\,\{A'\leftarrow B'\} \longleftarrow \{A''\leftarrow B''\}$$

and $$\{a\leftarrow b\}\circ\{a'\leftarrow b'\} = \{(a\circ a')\leftarrow(b'\circ b)\}\,\mathbf{:}\,\{A\leftarrow B\} \longleftarrow \{A''\leftarrow B''\}$$

1.2 Comments on pages

The comments on pages are in numerical order, but I got tired of fussing with counting lines and deciding how to count diagrams, footnotes, etc. I now use fractional page numbers. A page number may be followed by a decimal digit which is an estimate of how far down the page to look.

The symbol ::= means that the left side should be replaced by the right side, if the right side is in quotation marks, then it is more-or-less the suggested replacement, otherwise it is just a comment on what the replacement should be — and I should mention the symbol "—" for time delay (separates original comment from later addition).

1.2.0.1 Coend calculus, by Fosco Loregian [17]

These comments were orignally written in reference to version 5 (arXiv:1501.02503.pdf of 21-Dec-2019). In the process of putting them on the web site, I discovered that there is now a version 6 (arXiv:1501.02503.pdf of 11-Dec-2020). I made some updates, but no doubt some of these comments or page numbers are obsolete.

ii.1 (C2) introduces the integral notation for end, but (C1) did not introduce that notation for co-end, but it is used a few lines later.

ii.3 (C2) "an 'object of invariants' of 'fixed points' for the same action" ::= should that "of" be "or"?

ii.5 "a functor G", "a group G" ::= G changes its meaning rather abruptly.

ii.6 "the quotient of $X \times G$ for the equivalence relation $(g.x, y) \sim (x, g.y)$" ::= Should it be $X \times Y$? The equivalent pairs look like elements of that.

xi.6 It says: "the symbols $\_, =$ are used as placeholders".

The occurrence of an example of using more than one placeholder has vanished, or at least moved, in version 67 (v6), so this may be moot.

Would it not work as well to use single underscores in all cases? Can there be two occurrences of the same placeholder meaning two occurrences of the same variable? Does swapping the placeholders swap the order of the arguments?

Are single and double interchangable? Are (a) $\mathbf{C}(\_, =)$, (b) $\mathbf{C}(=, \_)$, (c) $\mathbf{C}(\_, \_)$, and (d) $\mathbf{C}(=, =)$ equivalent?

I understand the advantage of a single placeholder, but as soon as we need more than one, is it not time to use names (single letter variables)?

xi.8 "we denote $\Delta[n]$ the representable presheaf" but page 6.3: "a *wedge* for P is a dinatural transformation $\Delta_D \Rightarrow P$ from the constant functor" — So Δ has two meanings, one with subscript, one with [ ]. No problem.

— Except that it is a noun or expression that denotes a thing. "We" use a noun to denote. So ::= "we use $\Delta[n]$ to denote…" or
::= "we denote by $\Delta[n]$…" or
::= "we let C^B denote…" (as on page 3.2).

Or even better, leave us out of it! Whoever "we" are, we are not the subject of discussion so: ::= "$\Delta[n]$ denotes…".

There are many occurrences of this throughout the book, and of other expressions which are on the borderline between quaint phrasing and grammaticaal errors. I marked some of them, but in the future I will ignore them unless they cause real confusion (in me).

xi.8 —Typesetting error fixed in v6.

The use of a hiragana character for the Yoneda embedding is kind of cool, but I don't think I will be using it—I don't know how. What would Confucius do?

— Just for purposes of these notes (when quoting Loregian) can I say

$$£ : \mathbf{C} \to \mathbf{Cat}(\mathbf{C}^{op}, \mathbf{Set})$$

for the Yoneda embedding in the un-numbered(!) Lemma at beginning of §2.1 on page 33?
— I could make that look more like a hiragana yo by overprinting a little circle, but why bother?

— Mac Lane [18, §III.2(7), p62] uses Y'. Mac Lane and Moerdijk [19, §I.1(viii),p26] use $\mathbf{y}$. Jacobs [9, §7.3.12] uses $\mathcal{Y}$. Lambek and Scott [15, §O.2.9] use $\mathrm{Hom}^*_{\mathcal{A}^{op}}$. Johnstone [11, VI.1.1, p224] uses Y. — Check which of these are op-Yoneda.

xiii The sephirotic tree was cute, but gone in v6.

3.4 (1.2) "a natural transformations" ::= "a natural transformation" (not plural)

3.6 "mutely depending on the variable B in its codomain"
The variable B occurs twice in the domain and never in the codomain of $\epsilon_{X,(B)}$

—Also: what does "mutely" mean? In other places it seemed to mean what Mac Lane [18, §IX.4,p215] calls "dummy", that is, the functor ignores the variable, in that it factors through the projection from a product.

3.8 Is it not the case that figure (1.3) is a co-wedge? If I were writing this, I think I would use this to introduce co/wedges directly, as in section 1.3.2 on page 18, below, and then dinatural transformation can be explained (if at all) as combinations of wedges, natural transformations, and co-wedges. See the Hexagon of Transformation, figure 6 on page 31 below.

Since co/ends are co/terminal co/wedges, we want to describe co/wedges as quickly and simply as possible. Dinatural, supernatural, and extranatural are more complicated than co/wedges transformations.

4.3 It says "we have an equation", but then exhibits two, (1.5) and (1.6). The first (1.5) equates the top and right of the diagram (1.4) to the left and bottom. I haven't a clue what the second (1.6) is.

— Update a day or two later: I think (1.6) follows from (1.5) because $u \circ A = X^{B'} \circ u$ and $B' \circ f = f \circ B$. A half a line saying that would have been helpful; and maybe a reference to CWM [18, §IV.7.3].

— Update months later. Woot! It looks like I have accidentally done exercise 1.1. Is this the best way to arrange it?
"we have an equation" ::=
"we have two equations which are left to the reader to prove as Ex. 1.1".

4.4 Both equations involve X, B, and B', but suddenly comes a claim about C! I think C should be X. That also makes the claim match (1.3). — Fixed in v6.

5.1 Diagram (1.8) has lower case 'c's that should be upper case — Fixed in v6.

6.3 A wedge points *away* from the vertex. So does a cone.
Argument with references commented out.

7.8 See figure 3 for Loregian's (1.14).

8.8 It's not at all clear how the the three squares of (1.17) can be juxtaposed. In fact, I think it takes five squares, although three of them are instances of the first type. See the Hexagon of Transformation, figure 6 on page 31 below.

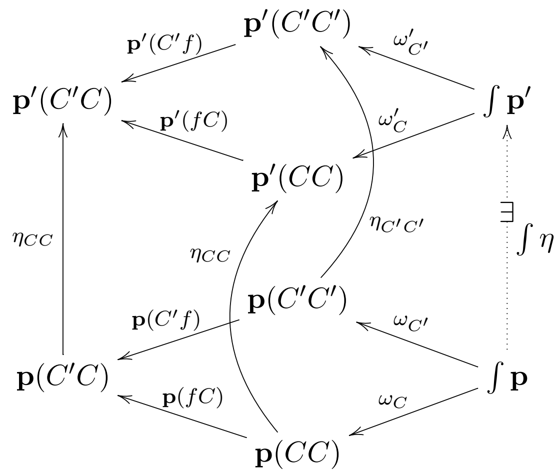


Figure 3: $\int$ is a functor

9.3 "It's worth to mention that ..." ::= "It's worth mentioning that ..." or "It's worthwhile to mention that"

"we are given arrows $F(B,B) \overset{\alpha_{BB'}}{\longrightarrow} G(B'B')$" – Definition 1.1.8 defines an extranatural transformation from P to Q as a triply indexed family of arrow. What is this doubly indexed arrow from F to G?
—It seems that $F = P$, or rather $F = \bar{P}$ were $\bar{P}$ to be defined, $G = \bar{Q}$, and $B' = C$. Or something.

9.8 Does this "graphical calculus" have any relation to Kelly-Mac Lane graphs [7, 4]?

11.0 Despite the rolling eyes symbol, I think I had better read Proposition 1.1.12 very carefully. What does it mean to say that P is a "particular kind" or "special case" of Q? Ordinarily I would expect it to mean that if any structure or object has property P then that *very same* object must necesssarily have property Q. For example, Abelian groups are a particular kinds of groups. For another example, one might say that rings are kinds of Abelian groups; not ones in which the group operation has a further property, but ones with a further operation. Neither of those examples seem to apply here.

Suppose things of type T can have properies D and E. We might prove that Es are particular kinds of Ds, as follows: Assume a thing t of type T; construct a $t' = f(t)$ also of type T, show that $D(t')$ implies $E(t)$. Does that work? Is it relevant? Is there a better way to summarize that proof in a theorem?

How does (1.20) relate to the hexagon (1.16) in the definition of Extramatural transformation, Definition 1.1.8? The functors F and G are not even of the form $T : \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B} \to \mathbf{E}$ and $S : \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{E}$, which is the precondition for Eilenberg and Kelly's definition of extranatural, and seems to match (1.16) on page 7 and Def 1.1.8 on page 6. of this book [17]. See Theorem 2.2 below.

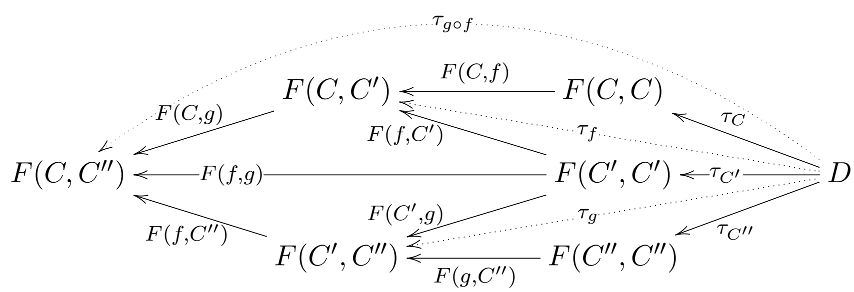
11.4 "Following the instructions… $G(A', A) = G(Y', Z', Z)$" ::= "$G'(A', A) = G(Y', Z', Z)$" (note G', not G).

11.8(1.20) It doesn't cure everything, but changing $G(1, h, 1)$ to $G(1, 1, h)$ on the label of the bottom arrow makes the types match better. Because Z' changes to Z in the third place; h is already in second place in rightmost vertical arrow.

11.9 "anonymous" ::= "eponymous" — fixed v6
"due to N. Yoneda, which" ::= "due to N. Yoneda, who" — Not fixed.
If "which" refers to the notation, not to Yoneda, then "which is introduced in [Yon60] along with most of the notions"

12.8 "The variable of integration c [in $\int_C F$] appears twice" ::= No it doesn't. Actually Mac Lane [18, §IX.5, p.219] writes $\int\limits_c S(c, c)$, an expression in which c occurs twice after the integral sign.

13.8 (1.22)



I understand it better after I draw it this way. That is no doubt partly because I had to think about it to re-draw it. This aligns similar objects in columns, makes some similar arrows parallel, removes the bogus three dimensional appearance, and labels the unlabeled arrow $F(C', g)$. This is a diagram in $\mathbf{D}$ and assumes $F : \mathbf{D} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$ and $\tau : F \Leftarrow D$ and $C'' \xleftarrow{g} C' \xleftarrow{f} C$. See section 1.3.3.

14.3 "(of course, this will not be $\mho$-small if $\mathcal{C}$ was only $\mho^+$-small)" ::= "of course" is a too short proof. Is $\mho$-small defined somewhere? If $\mho$ is a cardinal (or some other kind of number) and $\mho^+$ is a larger number then it seems not even true. For example, it is *not* true that a set is not finite if it is only countable. Maybe it should say that it is not the case that for every $\mathcal{C}$, $\text{TW}(\mathcal{C})$ is $\mho$-small if $\mathcal{C}$ is $\mho^+$-small (but it might be for some $\mathcal{C}$). Given that objects of one category are arrows of the other, it seems that the distiction between small and locally small would be relevant.

15.5 "$\prod\limits_{\varphi : C \to C'}$ … where the product over $f : C \to C'$ can be expressed" ::= $f = \varphi$?
— Not fixed. The second product in (1.27) is over φ, but the following explanation (in parentheses) says "over the arrows f.

15.6 "the arrows F^*, F_* are easily obtained from the arrows whose $(f; C, C')$-components are" ::= Arrows don't have components. I think it means the indexed family of arrows. See section 1.3.5 below.

15.9(ft) Are we defining $\text{TW}^{op}(C) := (\text{TW}(C^{op}))^{op}$? By analogy with $cos^2(x)$, I would expect $\text{TW}^{op}(C) := (\text{TW}(C))^{op}$, but in that case we are missing an ^{op}. — Look hard at (1.26)! — It has a colimit over $(\text{TW}(C^{op}))^{op}$ so that is indeed the definition.

The notation suggests that there are four cases $(\text{TW}(C))^{op}$, $(\text{TW}(C^{op}))^{op}$, $\text{TW}(C^{op})$, and $\text{TW}(C)$, but it seems that there are really only two. The objects of $\text{TW}(C)$ or $\text{TW}^{op}(C)$ are just the arrows of $\mathbf{C}$; relabeling target and source makes no difference. The arrows of either are pairs of arrows of $\mathbf{C}$ which make a commutative square with the composite of three sides equal to the fourth. The only choice is whether the single side is the target or the source. Is that right?

17.2 "the image of the initial wedge" ::= "co-wedge"??? — A co-end is an initial co-wedge, no?

17.5 Compare (1.32) here with (1) and (3) in CWM [18, §V.4, p113]. They agree, in slightly different notation.

17.7 "edges all the arrows $C^{\S} \to f$ and $C'^{\S} \to f^{\S}$, as ..." ::=
"edges all the arrows $C^{\S} \to f^{\S}$ and $C'^{\S} \to f^{\S}$, as ..." — fixed in v6

18.2 "wedges for F and ones for $F^{\S}$" ::= "and cones for" — not fixed

18.8 (1.35) The left column of the left square says "$C = C'$, which is wrong.

21 "$\mathbf{D}(D', X \pitchfork D)$ (1.40)" ::= "$\mathbf{D}(D, X \pitchfork D')$" — fixed in V6

23.3 I think (1.49) is important. Cf. CWM [18, 219.5] and Kelly [13, §2.1] and see section 1.3.4 below.

The theorem says "we have a canonical isomorphism", but what dees "canonical" mean? See section **??** below.

"(this is simply a rephrasing of the wedge condition)" But where can I find the un-rephrased wedge condition? — Remark 1.1.9, page 7, page 9.0

26.7 1.4 "accordingly to these rules" ::= "according to these rules"

27.4 1.5 "Prove that dinaturality is strictly more general than extranaturality, following this plot."

What does that mean? We can define dinatural in terms of extranatural, can we not? See Theorem 2.2 below. I think Loregian's definition of extranatural is the same as Eilenberg and Kelly's definition. Compare figure (1.16) and Definition 1.1.8 in Loregian [17, p8], with the Hexagon of figure 6 on page 31 of this document, which is a copy of figure (4) of Eilenberg and Kelly [7, p388].

Proposition 1.1.12 on page 11 says: "Extranatural transformations are particular kinds of dinatural transformations". Isn't this the same? If it is already proven, then what is the point of excercise 1.5? Is it strictness or the plot? Let's look at it.

"Let $\Delta[1] = \{0 \to 1\}$ be the 'generic arrow' category and
$S, T : \Delta[1]^{op} \times \Delta[1] \to \mathbf{Set}$ the functors respectively defined by

$$\begin{array}{ccccccccc} \{1\} & \xrightarrow{c_1} & \{1,2\} & (0,0) & \longrightarrow & (0,1) & \{1\} & \longrightarrow & \{1\} \\ \downarrow & S & \downarrow \sigma & \downarrow & & \downarrow & \downarrow & T & \downarrow c_2 \\ \{1\} & \xrightarrow[c_2]{} & \{1,2\} & (1,0) & \longrightarrow & (1,1) & \{1\} & \xrightarrow[c_2]{} & \{1,2\} \end{array}$$ "

How do three squares define two functors? I think the center diagram is meant to be $\Delta[1]^{op} \times \Delta[1]$, and indeed the objects of that category are pairs $\{(x, y)|x, y \in \{0, 1\}\}$, but the arrows of $\Delta[1]^{op}$ go in the opposite direction. The only non-identity in $\Delta[1]^{op}$ is: $0 \leftarrow 1$, so the diagram should be:

$$\begin{array}{ccccccccc} \{1\} & \xrightarrow{c_1} & \{1,2\} & (1,0) & \xrightarrow{(1,\to)} & (1,1) & \{1\} & \longrightarrow & \{1\} \\ \downarrow & S & \downarrow \sigma & (\leftarrow,0)\downarrow & & \downarrow(\leftarrow,1) & \downarrow & T & \downarrow c_2 \\ \{1\} & \xrightarrow[c_2]{} & \{1,2\} & (0,0) & \xrightarrow[(0,\to)]{} & (0,1) & \{1\} & \xrightarrow[c_2]{} & \{1,2\} \end{array}$$

The left and right diagrams are the images in $\mathbf{Set}$ of the middle diagram *via* the functors. I haven't finished the exercise, so I am not sure that works.

I thought I found the opposite claim in my old notes. I said that extranatuality is more general that dinaturality. See Theorem 2.2 on page 32 below. If it were not for this exercise I would think that together we have proven that the are equivalent. Instead, I thought that the exercise must show that a particular way of constructing a dinatrural transformation given an extranatural does not work. But after I finished my proof I found the construction ruled out by the exercise is almost exactly the one I used. I remain confused.

27.7 (last line of Ex.1.5) "which is not extranatural when in 1.1.8 we choose $\mathbf{A} = *$ and $\mathbf{B} = \mathbf{C}^{op} = \mathbf{\Delta}$"
Since "$S, T : \Delta[1]^{op} \times \Delta[1] \to \mathbf{Set}$" and we don't want the answer to the exercise to be $\Delta[1] \neq \mathbf{\Delta}$, there must be some way to extend S and T to all of $\mathbf{\Delta}$. Either that, or I am totally wrong about how the three squares define two functors.

30.5 $\left(\int_C \mathbf{C}(C, C)\right) \otimes \mathbf{D})$ parentheses don't balance!
—Ooops! Yes they do, the last paren is not part of the formula, but part of the English: "(parenthesisation is important:. . . ". Maybe this could be rearranged to keep the English paren from getting tangled into the math.

30.8 (last line of last bullet point) "(o equivalently" ::= "(or equivalently"

32.8 "Along the present chapter" — "Along" ::= In, Within, Throughout
"an an object of a 0-cell $A \in \mathsf{Cat}$" – (note: a 0-cell in Cat is a category)

34.1 "a pair $\langle u, \eta\rangle$ *exhibits* the left extension $\mathrm{lan}_g f$" ::= "a pair $\langle u, \eta\rangle$ *exhibits* the left Kan extension $u = \mathrm{lan}_g f$"
To say what *exhibits* the Kan extension leaves the question what *is* it? I think it's u. Maybe it's $\langle u, \eta\rangle$, but then why say it exhibits the extension rather than it *is* the extension? Why

is $\mathrm{lan}_g f$ lower case here? It is $\mathrm{Lan}_G F$ all through § 2.3 starting on page 37.9. Or is one the extension and one the Kan extension?

($\bar{\alpha} * g$ is the whiskering of A.2.3)::= "$*$" is whiskering, but wnat is the bar over the α?

38.6 "$\mathcal{L}^{\vee}(C) \Rightarrow F$ is in bijection..." ::= what's that $\vee$ superscript?
— It shows up again on page 41.1 where it means vector space dual. Later on the same page it is called an involution. But what *is* it?
— and in (2.25) on page 42. — Example A.3.3 page 232. — Formula (2.36) on page 49.
— page 48.5 it appears to call this a "duality involution".

— So is $[\,]^{\vee}$ the dualitity involution for the Yoneda emedding
$\mathcal{L} : \mathbf{C} \to \mathbf{Cat}(\mathbf{C}^{op}, \mathbf{Set})$ and $\mathcal{L}^{\vee} : \mathbf{C^{op}} \to \mathbf{Cat}(\mathbf{C}, \mathbf{Set})$? (Just a wild guess!)

— Unfortunately for that guess, Remark A.5.5, page 267.2 says: "$\mathcal{J} : \mathbf{C}^{op} \to \mathbf{Cat}(A, -)$

39.2 The right sides of (2.15) and (2.16) are the same except that C and C' are swapped. It would be clearer if variables were chosen so that they were the same, that is V and C in contravariant, and C' covariant position in both cases. I think (1.40) on page 19 is confused because of this.

40.8 "**Axiom** ... *Proof*" ::= Why are we proving an axiom? It seems that we are actually proving the theorem "There is a Yoneda structure on $\mathbf{Cat}$.", but what is it? Where are the definitions of $\mathbf{P}$ and χ^f in $\mathbf{Cat}$? What is an admissible functor?

58.0 "E1)" a strange way to write the transpose. — Neatly fixed in v6.

Riehl [21, p3,ft2]says: the $\#$ means "the map that is the left adjunct to." Too bad that's the opposite of a xylophone or piano keyboard, which has sharp to the right, flat left. Whatever.

58.4 "E4)... expliciting all its components," ::= "making all its components explicit" ("explicit" is an adjective, not a verb) — Unless it should be "explicating".

71.5 (2.5) "Use equations... that $\mathrm{Lan}_{\mathrm{id}}$ ... as expected" ::= "**to show** that ... as expected"

76.7 YI2) What is "essential image"? See Section 1.3.1 below.

88.9 Example 3.2.16, second paragraph, "Such... is intertwined..., is contained...." This sentence has two verbs (both "is"). Change comma to period, make second sentence: "A proof is in the appendix of [MLM92].

91.3 Example 3.2.18: a (Bénabou) cosmos must be **closed** but this is not stated in the example. Instead, we fitst learn of that requirement in **Notation** section of page 97.9.

92 Exercises 3.4 and 3.6 have downcast eyes, — fixed v6

97.3 "the notion of *cone* for a functor $F : \mathcal{J} \to \mathcal{C} \ldots F : \mathcal{J} \to \mathcal{B}$"
What is the point of changing $\mathcal{C}$ to $\mathcal{B}$? — fixed v6

97.5 "the terminal such diagram constitutes the ... limit (and dually, the initial object is the ... colimit)" In the ordinary, unenriched, case these are not terminal and initial in the same category, we must also change cones to co-cones. Don't we need to do something similar here?
— Yes. In formulæ (4.4) and (4.6) on page 99 we must change $\mathcal{C}(C, F(-))$ to $\mathcal{C}(F(-), C)$.

98.5 "where 1 is a shorthand" Should * be 1? — fixed v6

"and $\mathbf{C}(C, F(\_))$ is the functor $\mathbf{J} \to \mathbf{C}$ sending J to $\mathcal{J}(C, FJ)$" — $\mathcal{J}(C, FJ)$ should be $\mathbf{C}(C, FJ)$

In (4.2) the third occurrence of $\mathcal{J}$ should be $\mathcal{C}$!

99.5 Note.4.1.3 "is meant to evoke a THC situation" It seems THC stands for Tensor-Hom-Cotensor, not TetraHydroCannabinol. Might be worth spelling it out once. I did a linear search back to Ex.3.6 on page 92.

99.6 Example 4.1.4: Why does it define (or at least describe) $\lceil f \rceil$ and then never use it? Should some occurrence of f be $\lceil f \rceil$? Which one?

"a natural transformation $W \Rightarrow \mathbf{C}(C, f)$" ::=
A natural transformation is between parallel functors. Since $W : [1] \to \mathbf{Set}$, I must find a way to believe $\mathbf{C}(C, f) : [1] \to \mathbf{Set}$. I think I know $\mathbf{C}(C, f) : \mathbf{C}(C, X) \to \mathbf{C}(C, Y)$. How are those compatible? Does it help to write
"a natural transformation $W \Rightarrow \mathbf{C}(C, \lceil f \rceil)$" ?
It seems not, since now I need to think about $\mathbf{C}(C, [1])$ and $\mathbf{C}(C, \mathbf{C})$. (Because $\lceil f \rceil : [1[\to \mathbf{C}$.)

– I compare Loregian's (4.3) and Kelly's [14, (2.1)] and [13, (3.1)]. Now I think that $\mathbf{C}(C, f)$ is not the result of applying the functor $\mathbf{C}(C, [\,])$ to the arrow $f : X \to Y$ (in $\mathbf{C}$), it is the functor $\mathbf{C}(C, \lceil f \rceil`[\,]) : [1] \to \mathbf{Set}$. (or write $\mathbf{C}(C, \lceil f \rceil(-)) : [1] \to \mathbf{Set}$)

"$W1 \to \mathbf{C}(C', Y)$" ::= what is C'? should be just C. — fixed v6

"the universal property for $\mathrm{colim}^W f$" — ::= "$\lim^W f$" fixed v6

Page 271.9, A.6.13(iii) "where $\lceil W \rceil : \{*\} \to [\mathbf{C}, \mathbf{Set}]$ is the name of the functor W" ::= So is $[1] = \{*\}$? No. Because [1] is the generic arrow category, while $\{*\}$ is a one element set.

100.2 $\int_{J \in \mathcal{J}} \mathbf{Set}(WC, \mathbf{C}(C, FJ))$ ::= $\int_{J \in \mathcal{J}} \mathbf{Set}(WJ, \mathbf{C}(C, FJ))$ — fixed v6

102.4 Suddenly come two new symbols, ⨛ and Σ. — C ⨛ F is in the v6 index, 101, 105, 267. Σ is not there, 101 should be 102. 105 ::= 106. 267 ::- 268 The actual definition of C ⨛ W is A.5.9 on page 268. Σ is defined in A.5.13.

I think many of the page numbers in the index are off by one (or more). See comments on pages 306–308 below.

— I just discovered that the LaTeX command `\rotatebox{⟨degrees⟩}{⟨symbol⟩}` can rotate a symbol. This will come in handy for Linear Logic: ⅋ ! In the present situation I can define ⨛ as $\backslash rotatebox\{20\}\{\$\backslash!\backslash int\$\}$.

109.5 "$ph = q$" ::= What is "q"? Is it an unlabeled upper left arrow in (4.36)?

129.5,PN3 That's interesting. I have encountered the term "distributor" before and was puzzled about what it had to do with the distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$.

162.3 "objects of a monoidal category whose objects are natural numbers" ::= "Objects of a thing whose objects are numbers" is a long-windecd way to say "numbers". I think that's not what is meant. Should it say "indexed by" or "enriched over" or something?

— Maybe it is what is meant, except the operad is not the collection. "An operad, O, is a monoidal category whose objects are natural numbers, $O(n)$, and whose maps are

$$O(f) : O(n_1) \otimes \ldots \otimes O(n_k) \to O\left(\Sigma n_i\right)$$

satisfying suitable axioms", but now I wonder: what is f? I suppose it's some kind of label for an arrow, maybe a permutation and a label. Loregian does not, in this summarary, label the arrows. Maybe it's best to leave it out. I won't figure this out until somebody tells me what is $\otimes$, the tensor product of integers.

But it's evil to insist that the objects of an operad must be integers, because then there would be pairs of equivalent categories one of which is an operad, the other not, just because its objects are finite sets of some other type.

212–218(v5) The section §8.3 *Coends in Haskell* has been erased in v6. I'm a programmer; I had better write it myself. Except I am interested in co/ends in the hope that they may be useful to specify the semantics of `Scheme` macros.

242.3 "an infix semicolon as in $g; f$" — fixed in v6.

250.3 (A.15) may be the root of backward. Composition of functors is left following right, horizontal composition is left before tight. A poke in the eye to Kelly [13] and Mac Lane [18, §II.5].

There is nothing logically wrong with defining $A \boxminus B = B * A$, as long as there is no confusion, but why?

The next two items refer to the proof of associativity and accompanying diagrams, which have been deleted entirely in v6. That removes the obvious contradiction, but does it fix the problem of backward notation, or just hide it?

235.6(v5) (A.20) Here things go wrong. Although the definition of "$\boxminus$" in (A.18) (A.19) shows α left of β in both picture and formula, (A.20) and the following formula have μ, κ, and β in opposite order.

236.2(v5) "horizontal composition with an identity natural transformation coincides with whiskering: $\alpha \boxminus \mathrm{id}_H = H * \alpha$ and $\mathrm{id}_K \boxminus \beta = \beta * K$" ::= Something is backward. It's not clear what are the types of α, β, H, and K. Assuming they are as in (A.18), then it should be $\mathrm{id}_H \boxminus \alpha = H * \alpha$ and $\beta \boxminus \mathrm{id}_F = \beta * F$. In fact, if we agree that $\mathrm{id}_H = H$ and $\mathrm{id}_F = F$ then $\boxminus = *$. — This comment was written before the preceding two. See "Whiskering" in sectioh 1.3.1, on page 16 below.

250.5 (A.2..6) "$\xi : F \circ G \to \mathrm{id}_{\mathbf{C}}$ e $\eta : G \circ F \Rightarrow \mathrm{id}_{\mathbf{D}}$"
What is "e"? is that Italian for "and"? Why are the arrows different, rather than both $\to$ or both $\Rightarrow$?

237.5(v5) "an equivalence of categories is already able to preserve all isomorphism-invariant properties" ::= Surely this is false — and gone in v6.

250.6 (A.2.7)"with the only functor" ::= "with only the functor".

251 A.2.1 "there is plenty of ways" ::= "there *are* plenty of ways"

254 "Observe tat the following propeerty" ::= "Observe **that**".

255 A.3.1"*final* o *terminal*" ::="*final* or *terminal*"

256.3 "Extensions ... if we add a terminal object, and ... if we add an initial object, of co/cones over $\mathcal{I}$." This wording leaves me wondering which is the cone and which the cocone. See remark on page 4.6 above.

256 A.3.4"and is empty otherwise",
not strictly true since there is an identity on ∞ so $\mathcal{J}^{\triangleright}(\infty, \infty) \neq \emptyset$.

Now we have "right cones" and "left cones". Which are the co-cones? I think a co-cone is the image of a right cone; a cone is the image of a left cone. — That seems to be confirmed by Definition A.4.7 on page 244.9.

Mac Lane (CWM [18, §III.3, p.67]), speaks of cones under the base (which are cones from the base functor (the diagram)), and cones over the base [18, §III.4, p.71]). To visualize this, imagine the arrows pointing mostly downward. So Mac Lane's cones under the base have arrows to the vertex from the base. In other words they are Loregian's co-cones. Mac Lane's cones over the base are Loregian's cones. See pictures on page 24.

257.3 "the value of $\bar{D}$ on $-\infty$ is called the base of the cone" — Surely that is wrong! The base is the wide part (the diagram); the tip is the vertex (constant, diagonal). The difference between cone and co-cone is that the arrows of co-cone go to the vertex from the base, but the arrows of a cone go to the base from the vertex.

On the next page [246.5(v5)] "a co/cone amounts to a natural family of maps from a constant object (the base) or to a constant object (the tip)." — This seems to explicitly say the opposite of what I just said. Maybe there is some reason for what seems to be perverse terminology. — ::= "a co/cone amounts to a natural family of maps from/to a diagram, (the base), to/from a constant object (the tip or vertex)"

It says: "cones for D are exactly cocones for the opposite functor D^{op}" — If $D : \mathcal{J} \to \mathbf{C}$, then $D^{op} : \mathcal{J}^{op} \to \mathbf{C}^{op}$, (not $D^{op} : \mathbf{C} \to \mathcal{J}$ and not $D^{op} : \mathcal{J}^{op} \to \mathbf{C}$). But what is wanted is contravariant functor $\bar{D} : \mathcal{J}^{op} \to \mathbf{C}$. This follows the notation and terminology of Mac Lane (CWM [18, §II.2,p.33]), or at least I'm trying.

247.3(v5) "Let $J = \emptyset$" ::= "Let $\mathcal{J} = \emptyset$" (wrong font, — gone in v6

248.5(v5) "there is an cone" — This entire section of Examples seems to be gone in v6, at least I can't find it.

255.0(v5) "in some places, final functors are called cofinal and viceversa."

Loregian makes this claim in a footnote. No references are given; I can't check it.

This entire section seems to have been removed in (v6), in which case this whole comment is obsolete,

Mac Lane [18, §IX.3] defines a variant of what the set theorists call a "cofinal function" but calls it a "final functor", dropping the "co" as not related to dualization. In the context of set theory "cofinal" means "final together with".

Levy [16, Ch.II, Def.1.16] says:

> Let $\langle A, R\rangle$ be an ordered class, and let B be a subclass of A. B is said to be *cofinal* in $\langle A, R\rangle$ if for every member $x \in A$ there is a member $y \in B$ which is greater than or equal to x. (In symbols: $(\forall x \in A)(x \in B \vee (\exists y \in B)xRy)$.)

For Levy, $\langle A, R\rangle$ is an ordered class iff R is an irrreflexive and transitive relation on A.

Loregian proposes to rename Mac Lane's "final functor" to "cofinal functor". The definition of cofinal now seems to better match the definition of cofinal used by set theorists. That seems good. I now find two occurrences of the word "cofinal":

page 286 "every filtered category $\mathcal{A}$ admits a cofinal functor from an ordinal α_A."
page 18 "There is a cofinal functor ϑ from $\mathbf{C}^\S$ to $\text{TW}(\mathbf{C}^{op})^{op}$

Cofinal is not in the index. What does it mean today?

265(v5) "A presheaf is a functor $F : \mathbf{C}^{op} \to \mathbf{Set}$" this seems to say, what I have always understood, that a presheaf is contravariant. On page 266.6: "We call a presheaf F representable if it is isomorphic to $\mathbf{C}(\_, X)$ or $\mathbf{C}(X, \_)$ (depending on its variance), for some object $X \in \mathbf{C}$." That seems to say that a presheaf can have either variance. I have no objection to that expanded usage, but it should be mentioned when introducing the term.

The section seems to have been deleted. The word "presheaf" is still used, but it's not in the index, and I have not found a definitiom.

265 (A.5.2) "sending $A \in \mathbf{C}$ into the set of morphisms $u : A \to X$" ::= "the set of morphisms $\mathbf{C}(A, X)$", because u is not a set, but a typical member of the set. — One might also say: "the set of morphisms, u, of the form $u : A \to X$"

"a functor, called the presheaf associated to X" ::= This is a bit muddled. It doesn't quite finish defining the presheaf functor before starting on the association functor. Should say: the action of $\mathcal{C}(\_, X)$ on an arrow $a : B \to A$ is given by the function that takes $u \in \mathcal{C}(A, X)$ to $\mathcal{C}(a, X) = u \circ a \in \mathcal{C}(B, X)$. Furthermore, this association can itself be made into a functor which sends $f : X \to Y$ to the natural transformation $\mathcal{C}(\_, X) \Rightarrow \mathcal{C}(\_, Y)$ from the presheaf functor associated to X to that associated to Y having components

$$f_{*,A} : \mathcal{C}(A, X) \to \mathcal{C}(A, Y) : u \mapsto fu$$

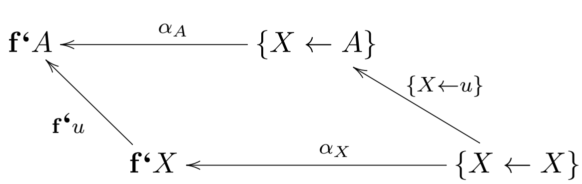
"the function that sends $f : X \to Y$ in the natural transformation" ::= "… sends … **to** the natural transformation"

265.9 The step $\alpha_A(\mathrm{id}_X \circ u) = Fu \circ \alpha_X(\mathrm{id}_X)$ is not clear, partly because the second term looks like $Fu \circ (\alpha_X(\mathrm{id}_X))$, which does not work.

So, writing $\{X \leftarrow [\,]\}$ for what the book calls \hirigana{yo}X:

Theorem 1.1. *Let* α, β $\mathbf{::}\, \mathbf{f} \Longleftarrow \mathbf{C}\{X \leftarrow [\,]\}$ $\mathbf{:Set} \longleftarrow \mathbf{C}^{op}$. *That is,* α, β *are natural transformations between pre-sheaves (contravariant set valued functors).*
If $\alpha_X‘X = \beta_X‘X$ *then* $\forall u : X \leftarrow A\,\mathbf{.}\; \alpha_X‘u = \beta_X‘u$.

Proof.



$$\begin{aligned} \alpha_A‘(u) &= \alpha_A‘(\mathrm{id}_X \circ u) && (1)\\ &= \alpha_A‘(\{X \leftarrow u\}‘(\mathrm{id}_X)) && (2)\\ &= (\alpha_A \circ \{X \leftarrow u\})‘(\mathrm{id}_X) && (3)\\ &= (\mathbf{f}‘u \circ \alpha_X)‘(\mathrm{id}_X) && (4)\\ &= (\mathbf{f}‘u)‘(\alpha_X‘(\mathrm{id}_X)) && (5)\\ &= (\mathbf{f}‘u)‘(\beta_X‘(\mathrm{id}_X)) && (6)\\ &= \beta_A‘(u) && (7) \end{aligned}$$

Equation (4) holds because α is natural, as shown in the diagram. Equation (7) is justified by repeating all previous steps in reverse order, with β substituted for α. □

266.5 "id_X can be thought as the *universal element*" — CWM [18, p.57] defines universal element; is this one? See section 1.3.6 on page 26 below.

"we call t a universal element if $\hat{t}_X$ is invertible" ::= Does it really mean "$\hat{t}_X$ is invertible" itself, or the process $t \mapsto \hat{t}_X$ is invertible? — and what does that hat (ˆ) mean? There is no hat on anything in the preceding discussion of the Yoneda lamma. — It shows up again in Lemma A.6.8 on the following page. I had better read that harder.

— Why are both t and id_X called universal elements?

Is $(\hat{t}_X)_A = \alpha^x_A$ in the notation of the preceding proof that Y is surjective? (substituting $t = x \in FX$) — Because Y has been proven to be bijective, is it not just $(\hat{t}_X) = Y^{-1}(t)$?

269.7 "[Grothendieck] is equally important than the Yoneda lemma" ::= "is equally as important as the Yoneda lemma"

268.9 There is a double shaft arrow inside the square. What does that mean? Is it a natural transformation? Does the square not strictly commute? — Also the text of (iii) says $\lceil W \rceil : \{*\} \longrightarrow [\mathbf{C}, \mathbf{Set}\}$ but the diagram shows $* \xrightarrow{\lceil W \rceil} \mathbf{Cat}(\mathbf{C}, \mathbf{Set})$. Kelly [13, §1.6] says: "$\mathcal{V}(X, Y)$ is $[X, Y]$", so if $\mathcal{V} = \mathbf{Cat}$ then the targets match, but are we to believe $* = \{*\}$? — The left vertical side of the rectangle is a bit mysterious.

270.3 "it is equally easy to see that a morphism $f : X \to X'$ induces a function of sets between the fibers $G^{\leftarrow}X'$ and $G^{\leftarrow}X$ (see Figure A.1 below);" Another surprise notation. I know what I would mean if I wrote $G^{\leftarrow}$", (functor category from two object, one arrow category) what does it mean here? — I think this *is* the definition disguised as a use. $G^{\leftarrow}X$ is the fiber over X; but it never just says that.

— I don't find it in the index.

— on page 23(v5) "the comma category $G^{\leftarrow} = (\mathrm{id}_{\mathbf{C}} \downarrow \mathrm{id}_{\mathbf{C}}) = \mathbf{Cat}([1], \mathbf{C})$ (see A.3.20 and A.10.1)" Note: A.3.20 is on page 240.8; A.10.1 on page 259.1. They define comma category, and simplex category, respectively.

272.0 The section is called "Monoidal categories and monads" and the first thing it says is: "Let $\mathbf{C}$ be a monoidal category" but the definition of "monoidal category" has gone missing. Maybe every child already knows that, since both Mac Lane and Kelly have defined it and I think they agree.

Definition 1.2. *A monoidal category is a category equipped with an associative and unital, bi-valent endofunctor. See Mac Lane [18], Kelly [13] . . .*

288.1 "we wer secretly talking" ::= "were"

295 In Fubini rule, what is the type of T? If $T : \mathbf{C} \times \mathbf{D} \times \mathbf{C}^{op} \times \mathbf{D}^{op} \to \mathbf{A}$ then the first two integrals are right, but if $T : \mathbf{C} \times \mathbf{C}^{op} \times \mathbf{D} \times \mathbf{D}^{op} \to \mathbf{A}$ then the third integral is right. On page 20, formula (1.40), all three occurrences of the functor F have arguments in the same order.

303 reference [Kel82] What's a CUP archive? See Kelly [13].

305 reference [RV17b] gives page numbers, but not Journal or Publisher.

306 The index is new in v6. The entry for $(\_)^\sharp$ and $(\_)^\flat$ refers to 202, but I think I see them first on page 204.

308 The entry for "Einstein notation" refers to page 132, but I see:
"**Notation 5.1.10** (Einstein Notation)" on page 133. Is this a clerical error, a software problem, or an odd choice?

1.3 Longer comments

This section contains comments which are longer and more general. If they are related to a specific page, definition, or formula, that is noted at the far right.

1.3.1 Definitions

References go here:

When I encounter an unfamiliar term, I make a note on the page where I saw it and refer to this list, most of which is taken from either wikipedia or nLab.

essential image used 46.3, 76.7, 267.4, 292.9:

The essential image of a functor $F : A \to B$ between categories is the smallest replete subcategory of the target category B containing the image of F. The image (*simplicitur*) is the smallest subcategory which contains all the arrows which are strictly the images of arrows in A. A subcategory D of C is replete if for any object x in D and any isomorphism $f : x \cong y$ in C, both y and f are also in D.
Equivalently: the inclusion $D \hookrightarrow C$ is an isofibration.

isofibration used no where in v6:

An isofibration is a functor $p : E \to B$ such that for any object e of E and any isomorphism $\phi : p(e) \cong b$, there exists an isomorphism $\psi : e \cong e'$ such that $p(\psi) = \phi$.

discrete fibration Definition A.5.12:

The definition of a discrete fibration, requires for any arrow k in the base category there is a *unique* arrow lying over k. The definition of a fibration requires an arrow lying prone over k. Can we prove that if there is a unique arrow lying over k then it lies prone? It seems we need this in order to prove that a discrete fibration is a fibration, which I like to hope is true.

(right—left) whiskering defined A.2.3:

Seems to be just a goofy word for horizontal composition of a 1-cell with a 2-cell.

— It's in the index, "Whiskering, 246". It's actually Definition A.2.3 on page 248.8.

— Does this happen automatcally if an n-cell equals the identity $n+1$-cell on it?

— The book "*Homotopy Type Theory* " [24] also uses (and defines) this term. It may be a popular word, whether or not goofy.

— Loregian [17, Def.A.3.7] *defines* horizontal composition in terms of whiskering, Does this work in the general case of enriched categories?

— I start with $\mathbf{Cat}$-enriched category theory and think of horizontal composition as a bivalent functor on $\hom$ objects, which is assumed given as part of the structure of a 2-category. There seems to be another approach here. Look at the Eckmann-Hilton argument. CWM [18, II.5.Ex.5, p45]; HoTT [24, Thm.2.1.6]

— I'm confused. See notes on pages (v5) 236–237. Do we need to prove that horizontal compostion is commutative, or just decide on a direction?

canonical used 23, Thm.1.4.1:

Is "canonical" a real technical term? It seems to mean that there is one preferable way to make a construction, but does it imply that there is no other way to do it?

There is a canonical way to make a product of sets by taking ordered pairs, but that doesn't imply that there is no other product. In any case, there are several ways to define "ordered pair" in pure set theory.

Theorem 1.4.1 is weak unless "canonical" means something strong.

— Maybe canonical is a way to smuggle constructive content into an argument in non-constructive (classical) logic. In fact, any set of the approriate cardinality could be the cartesian product, if the projections π_0 and π_1 are defined to match. A thing is canonical if it has been constucted in a way that works in all cases and that provides enough information to construct the required relationships to other things. See [24].

— Are canonical morphisms automatically coherent?

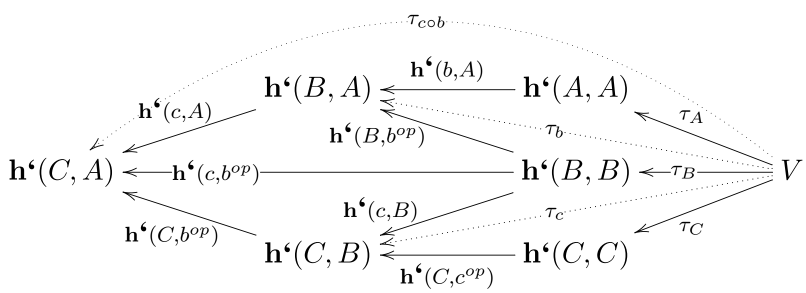
1.3.2 Wedges

Def.1.1.4, p.6:

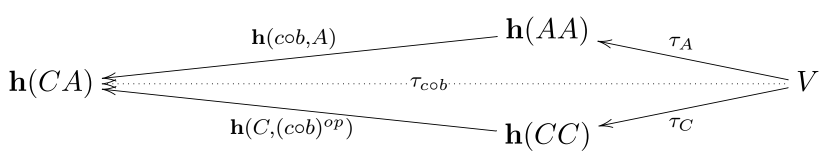
Definition 1.3. *Given categories* $\mathbf{C}$ *and* $\mathbf{V}$*, with a bivariant functor* $\mathbf{h} : \mathbf{V} \longleftarrow \mathbf{C} \times \mathbf{C}^{op}$*, and an object* $V \in \mathbf{V}$*, we say that* τ *is a* <u>wedge</u> *(in* $\mathbf{V}$*, over* $\mathbf{h}$*, from vertex* V*), written* $\tau : \mathbf{h} \Leftarrow V$*, iff* τ *is a family of arrows indexed by elements of* $\mathbf{C}$ *that satisfies the* <u>wedge condition</u>*, which is that for each object* $A \in \mathbf{C}$*,* τ_A *is an arrow* $\tau_A : \mathbf{h}‘(A, A^{op}) \longleftarrow V$ (in $\mathbf{V}$)*, and for each arrow* $b : B \longleftarrow A$ (in $\mathbf{C}$)*,* τ_b *is a proof that* $\mathbf{h}‘(b, A^{op}) \circ \tau_A = \mathbf{h}‘(B, b^{op}) \circ \tau_B$ *(or one could say that* $\tau_b = \mathbf{h}‘(b, A^{op}) \circ \tau_A = \mathbf{h}‘(B, b^{op}) \circ \tau_B$ *is the common value). Each of the* τ_A *and* τ_b *is called a* <u>component</u> *of the wedge.*

The following is the diagram (1.22) of Loregian [17, p.12], redrawn for the current context. It shows a wedge $\tau : \mathbf{h} \Leftarrow V$ where $\mathbf{h} : \mathbf{V} \longleftarrow \mathbf{C} \times \mathbf{C}^{op}$ and $V \in \mathbf{V}$. Two components of the wedge are the parallelograms at the right with dotted diagonal arrows through them. There is one such component for each arrow of $\mathbf{C}$. Assume $b : B \longleftarrow A$ and $c : C \longleftarrow B$ (in $\mathbf{C}$).

Here I write $A = A^{op}$ for an object in the opposite category.



Because $\mathbf{h}‘(c, A) \circ \mathbf{h}‘(b, A) = \mathbf{h}‘(c \circ b, A)$ and $\mathbf{h}‘(C, b^{op}) \circ \mathbf{h}‘(C, c^{op}) = \mathbf{h}‘(C, (c \circ b)^{op})$ the outer perimeter is also a component, as shown below, omitting apostrophes and commas between single letters. Note that $\tau_{c \circ b} = \mathbf{h}(c, b^{op}) \circ \tau_B$; to reduce clutter this is not drawn below, but can be found amidst the clutter above.



Definition 1.4. *Given a bivariant functor* $\mathbf{h} : \mathbf{V} \longleftarrow \mathbf{C} \times \mathbf{C}^{op}$*, there is a* <u>category of wedges</u> *over* $\mathbf{h}$*, denoted by* $\mathrm{Wd}(\mathbf{h})$*, which has wedges over* $\mathbf{h}$ *as objects. Given two such objects,* $\tau : \mathbf{h} \Leftarrow V_0$*. and* $\sigma : \mathbf{h} \Leftarrow V_1$*., an arrow* $d : \tau \longleftarrow \sigma$ (in $\mathrm{Wd}(\mathbf{h})$) *is an arrow between their vertices* $v : V_0 \longleftarrow V_1$ (in $\mathbf{V}$) *such that* $\forall C \centerdot \tau_C \circ v = \sigma_C$*.*

Kelly [13, §2.1] uses somewhat different language, but he defines the end $\int_A T(A, A)$ in a way that I think is equivalent. He calls the terminal wedge the "counit", and defines it as a "universal $\mathcal{V}$-natural family".

1.3.3 Twisted Arrows

Page 14 Def.1.2.2:

Recall, for a category $\mathbf{C}$, the functor[2] $\mathbf{hom_C} \mathbin{:} \mathbf{Set} \longleftarrow \mathbf{C} \times \mathbf{C}^{op}$. When given two objects $A' \in \mathbf{C}$ and $A^{op} \in \mathbf{C}^{op}$, it yields the set of arrows

$$\mathbf{hom_C}`(A', A^{op}) = \mathbf{C}\{A' \leftarrow A\} = \{a | a \mathbin{:} A' \longleftarrow A \text{ (in } \mathbf{C})\}$$

and given two arrows $g \mathbin{:} B' \longleftarrow A'$ and $f \mathbin{:} A \longleftarrow B$ (in $\mathbf{C}$), (or $f^{op} \mathbin{:} B^{op} \longleftarrow A^{op}$ (in $\mathbf{C}^{op}$)) it yields the function of type

$$\mathbf{hom_C}`(g, f^{op}) = \mathbf{C}\{g \leftarrow f\} \mathbin{:} \{B' \leftarrow B\} \longleftarrow \{A' \leftarrow A\}$$

which yields

$$\mathbf{C}\{g \leftarrow f\}`a = g \circ a \circ f \in \{B' \leftarrow B\}$$

given $a \in \{A' \leftarrow A\}$.

The twisted arrow category, $\text{TW}(\mathbf{C})$, is unrelated[3] to the twist map, $\langle \pi_1, \pi_0 \rangle$, of Mac Lane and Moerdijk [19, §IV.8,p198], which swaps the factors of a product. Rather, $\text{TW}(\mathbf{C}) = (\mathbf{C} \times \mathbf{C}^{op}) \int \mathbf{hom_C}$ is the category of elements[4] of the $\mathbf{hom_C}$ functor. Its objects are pairs $((A', A^{op}), a)$ consisting of an object $(A', A^{op}) \in \mathbf{C} \times \mathbf{C}^{op}$, together with an element of the corresponding set $a \in \mathbf{hom_C}`(A', A^{op})$, that is, an arrow $a \mathbin{:} A' \longleftarrow A$ (in $\mathbf{C}$). Its arrows are

$$((g, f^{op}), P) \mathbin{:} ((B', B^{op}), b) \longleftarrow ((A', A^{op}), a) \text{ (in } \text{TW}(\mathbf{C}))$$

where $(g, f^{op}) \mathbin{:} (B', B^{op}) \longleftarrow (A', A^{op})$(in $\mathbf{C} \times \mathbf{C}^{op}$) (or equivalently, $g \mathbin{:} B' \longleftarrow A'$ (in $\mathbf{C}$) and $f \mathbin{:} A \longleftarrow B$ (in $\mathbf{C}$)) and P is a proof of the proposition that $g \circ a \circ f = b$. Note that, by definition, $g \circ a \circ f = (\mathbf{hom_C}`(g, f^{op}))`a = \mathbf{C}\{g \leftarrow f\}`a \mathbin{:} B' \longleftarrow B$ (in $\mathbf{C}$). If the proposition is false, then the proof P does not exist and so $((g, f), P) \mathbin{:} ((B', B^{op}), b) \longleftarrow ((A', A^{op}), a)$ is impossible. In that case, (g, f) still determines an arrow in the twisted arrow category, but its target is not b.

For reasons disscussed in **??**, objects and arrows should be the same type or "shape". The arrow $a \mathbin{:} A' \longleftarrow A$ in the second component of an object can be seen as a proof that $\exists a \mathbin{:} A' \longleftarrow A$ (in $\mathbf{C}$), or the proof P can be seen as an arrow in a category of commutative squares.

The official definitions of the objects and arrows of the twisted arrow category are rather heavy, so ... Let's write $A = A^{op}$ for an object (or identity) in the opposite category[5]. Sometimes, if $a \mathbin{:} A' \longleftarrow A$ is known, the object $((A', A), a)$ will be written as simply a, the target and source being implicit in the arrow. Also the arrow $((g, f^{op}), P)$ is written (g, f), with the proof P left as an exercise.

Define $\mathbf{D} \frown \mathbf{C} = \mathbf{D}^{\mathbf{C}}$ as an in-line notation for a functor category. There is a functor

$$\widetilde{[\,]} : \mathbf{D} \frown \text{TW}(\mathbf{C}) \longleftarrow \mathbf{D} \frown (\mathbf{C} \times \mathbf{C}^{op})$$

[2] Arguments are: covariant target on left, contravariant source on right.

[3] Maybe "unrelated" should be "related", see Remark 1 on page 21 below.

[4] This notation for the category of elements is used by Loregian [17, Def.A.6.11] who attibutes it to Gray [6].

[5] I am wish-washy on the question of whether exactly the same object can be in two different categories, but it seems harmless to simplify notation in this case. I still want to be a stickler for the distinction (if any) between f and f^{op} for non-identity arrows.

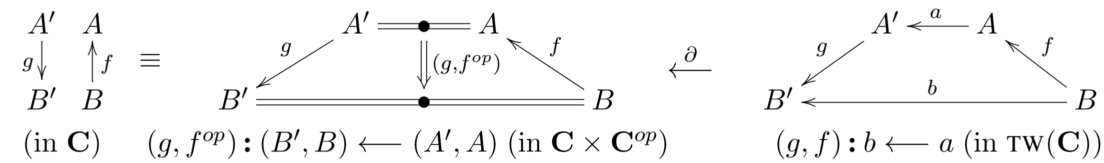


Figure 4: Construction of $\text{TW}(\mathbf{C})$

which takes $\mathbf{h} : \mathbf{D} \longleftarrow \mathbf{C} \times \mathbf{C}^{op}$ to a functor $\widetilde{\mathbf{h}} : \mathbf{D} \longleftarrow \text{TW}(\mathbf{C})$. This can be factored as $\widetilde{\mathbf{h}} = \mathbf{h} \circ \partial$ where $\partial : (\mathbf{C} \times \mathbf{C}^{op}) \longleftarrow \text{TW}(\mathbf{C})$ is a functor to be defined shortly.

In figure 4, double lines and circles in the middle diagram are meant to show the objects in $\mathbf{C} \times \mathbf{C}^{op}$, which are pairs of objects in $\mathbf{C}$. After application of $\mathbf{h}$ we get the same picture, only now the circles are objects of $\mathbf{D}$. If $\mathbf{D} = \mathbf{Set}$ then we can set $\mathbf{h} = \mathbf{hom}_{\mathbf{C}}$ and see the circles as sets of arrows.

I don't know if there is a "standard" symbol for the functor ∂ so, hearing no objection[6], I define:

Definition 1.5. *The functor $\partial : (\mathbf{C} \times \mathbf{C}^{op}) \longleftarrow \text{TW}(\mathbf{C})$, is given on objects of $\text{TW}(\mathbf{C})$, that is on arrows $a : A' \longleftarrow A$ (in $\mathbf{C}$), by $\partial‘a = (A', A^{op}) \in (\mathbf{C} \times \mathbf{C}^{op})$, and on arrows of $(g, f) : b \longleftarrow a$ (in $\text{TW}(\mathbf{C})$), by $\partial‘(g, f) = (g, f^{op}) \in \mathbf{C} \times \mathbf{C}^{op}$.*

It's almost true that $(\mathbf{C} \times \mathbf{C}^{op}) \supseteq \text{TW}(\mathbf{C})$ and ∂ is the injection of the subcategory. Several objects in $\text{TW}(\mathbf{C})$, e.g. $a_0 : A' \longleftarrow A$ and $a_1 : A' \longleftarrow A$, are taken by ∂ to the same one $\partial‘a_0 = \partial‘a_1 = (A', A)$. For this reason, ∂ is not the injection of a subcategory, but even though objects become identified, the arrows all survive individually and still compose the same way. (What, exactly, does that mean???)

Theorem 1.6 ([17], Rem.1.2.3, p.13)**.** *If $\tau : \mathbf{h} \Leftarrow D$ is a wedge where $\mathbf{h} : \mathbf{D} \longleftarrow \mathbf{C} \times \mathbf{C}^{op}$ and $D \in \mathbf{D}$, are as in the definition above, then the family $\{\tau_a : \widetilde{\mathbf{h}}‘a \longleftarrow D \mid \exists A', A \,.\, a : A' \longleftarrow A\}$ is a cone over the base functor $\widetilde{\mathbf{h}} = \mathbf{h} \circ \partial$. Conversely, if τ_a is such a cone then the family $\{\tau_A | A \in \mathbf{C}\}$ is a wedge to the base $\mathbf{h}$.*

This correspondence expands to an equivalence between the category, $\text{Cn}(\widetilde{\mathbf{h}})$, of cones over $\widetilde{\mathbf{h}}$ and the category, $\text{Wd}(\mathbf{h})$, of wedges over $\mathbf{h}$.

Proof. Figure 5 shows a wedge in the center together with two cones, which are the top and bottom triangles of the wedge.

Let's write ${}_\lambda D$ for the functor with value D when applied to any object, and the identity on D when applied to any arrow, ${}_\lambda D = \lambda x \,.\, D : \mathbf{D} \longleftarrow \text{TW}(\mathbf{C})$. The variable x is irrelevant, so the notation leaves it out. In the definition of a wedge, D is just an object, but the constant functor will be needed for a careful definition of a cone as a natural transformation between functors.

[6]After having written most of this section, I discovered CWM [18, §IX.6,Ex.3,p.223] and the web page
https://ncatlab.org/nlab/show/twisted+arrow+category
which call the functor K and π, respectively. I can't find any other use of either.

The symbols ∂_0 and ∂_1 have been used for source and target, so it makes some sense to use ∂ for both together. Also ∂ means "border" in algebraic topology, and the two ends of a line segment are its border.

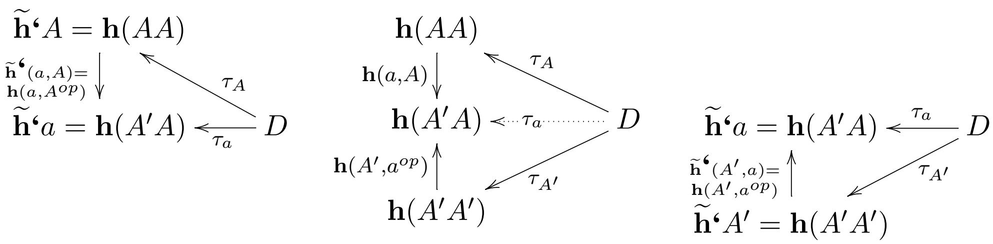
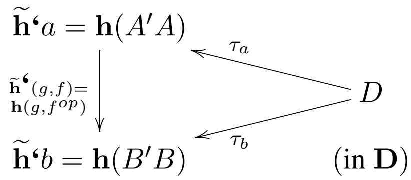


Figure 5: Wedges and Cones

Assume $\tau : \mathbf{h} \Leftarrow D$ is a wedge. To say that $\{\tau_a | a : A' \longleftarrow A\}$ is a cone over the base $\widetilde{\mathbf{h}}$ means that it is a natural transformation $\tau :: \widetilde{\mathbf{h}} \Longleftarrow {}_\lambda D : \mathbf{D} \longleftarrow \text{TW}(\mathbf{C})$. Spelling this out:



For each object of $\text{TW}(\mathbf{C})$, that is, for each arrow $a : A' \longleftarrow A$ (in $\mathbf{C}$) there is an arrow $\tau_a : \mathbf{h}(A', A) \longleftarrow D$ (in $\mathbf{D}$) and for every arrow $(g, f) : b \longleftarrow a$ (in $\text{TW}(\mathbf{C})$) the naturality condition $\widetilde{\mathbf{h}}‘(g, f) \circ \tau_a = \tau_b \circ {}_\lambda D‘(g, f)$ holds.

Since ${}_\lambda D‘(g, f)$ is just the identity on D, naturality means $\widetilde{\mathbf{h}}‘(g, f) \circ \tau_a = \tau_b$, which is proven by $\tau_b = \tau_{g \circ a \circ f} = \mathbf{h}((g \circ a), f^{op}) \circ \tau_A = \mathbf{h}(g, f^{op}) \circ \mathbf{h}(a, A) \circ \tau_A = \mathbf{h}(g, f^{op}) \circ \tau_a$

Conversely, assume that $\{\tau_a | a : A' \longleftarrow A\}$ is a cone to the base $\mathbf{h} \circ \partial$ from D . Let $A : A \longleftarrow A$ be the identity on A. Obviously $\tau_A : \mathbf{h}(A, A) \longleftarrow D$, For each $a : A' \longleftarrow A(\in \mathbf{C})$, we must prove the wedge condition $\mathbf{h}(a, A) \circ \tau_A = \mathbf{h}(A', a^{op}) \circ \tau_{A'}$.

Now $((a, A), S) : ((A', A), a) \longleftarrow ((A, A), A)$ (in $\text{TW}((\mathbf{C}))$, briefly $(a, A) : a \longleftarrow A$ is an arrow in the twisted arrow category, where S is any proof of $a \circ A \circ A = a$ which is trivia L. Since $\{\tau_a\}$ is a cone, the naturality condition is $\widetilde{\mathbf{h}}‘(a, A) \circ \tau_A = \tau_{A'}$. This shown in the left triangle in figure 5. Also $((A', a), S) : ((A', A), a) \longleftarrow ((A', A'), A')$ (in $\text{TW}((\mathbf{C}))$ is an arrow where S proves $A' \circ A' \circ a = a$ and the naturality condition is $\widetilde{\mathbf{h}}‘(A', a) \circ \tau_{A'} = \tau_a$. This is the right triangle. The middle diagram shows the two triangles pasted together along their common edge, and it is exactly the wedge condition. □

Remark 1. *The map $\langle \pi_0, \pi_1 \rangle$ twists a cartesian product, do we need a twist map on monoidal product to continue? We might need a twister to inhabit*

$$(\mathbb{A}\, t \,\blacksquare\, (\mathbb{A}\, C \,\blacksquare\, \alpha_C)‘C)‘t = (\mathbb{A}\, C \,\blacksquare\, (\mathbb{A}\, t \,\blacksquare\, \alpha_C)‘t)‘C$$

This called the combinator C *[23, p64] [8, p24,191] [2, §9.3],.*

The oppositely twisted arrow category $\text{TW}^{op}\mathbf{C}) = (\text{TW}(\mathbf{C}^{op}))^{op} =$

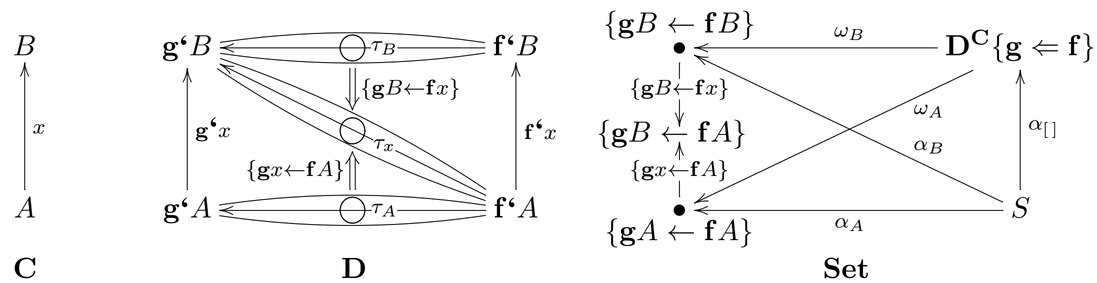


Figure 6: ω is a terminal wedge

1.3.4 Ends and Transformations

Page 23 (1.49):

Fix two categories $\mathbf{C}$ and $\mathbf{D}$, and two parallel functors $\mathbf{f}, \mathbf{g} : \mathbf{D} \longleftarrow \mathbf{C}$.

Two objects $A, B \in \mathbf{C}$ correspond to a set of arrows $\mathbf{D}\{\mathbf{g}‘B \leftarrow \mathbf{f}‘A\}$. Furthermore, letting A and B serve as parameters, this correspondence becomes a bivariant functor

$$\mathbf{D}\{\mathbf{g}‘[\,] \leftarrow \mathbf{f}‘[\,]\} : \mathbf{Set} \longleftarrow \mathbf{C} \times \mathbf{C}^{op}$$

Also there is a set of natural transformations $\mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\}$. This set is related to the functor by the formula in the following theorem. See also Mac Lane CWM [18, §IX.5,p.219], who briefly explains this formula, and Kelly [13, §2.1], who uses it to define enriched functor categories.

The symbol "$\cong$" means isomorphic, but iso in what category? That there is an isomorphism of sets just means that the sets are equinumerous, which is weak. Mac Lane says "$=$", which is surely not quite right. Loregian says it is a "canonical isomorphism of sets", but what does that mean? We need an isomorphism in the category of wedges over the bivariant functor that was displayed in the previous paragraph.

Apply the definition of a wedge over the bivariant functor $\mathbf{h} := \mathbf{hom_D}‘(\mathbf{g}‘[\,], \mathbf{f}^{op}‘[\,])$ in $\mathbf{V} := \mathbf{Set}$, with vertex $V := S$, (Def.1.4, p 18). This is shown in figure 6. The left side of the figure shows a typical arrow in $\mathbf{C}$. The functors $\mathbf{g}$ and $\mathbf{f}$ take it to arrows in $\mathbf{D}$, which are shown as the vertical sides of the rectangle in the middle. Ignoring the curved lines and circles, the rectangle is just what anyone would draw to explain what it means for τ to be a natural transformation, $\tau :: \mathbf{g} \Longleftarrow \mathbf{f} : \mathbf{D} \longleftarrow \mathbf{C}$.

The curved lines are meant to suggest other arrows parallel to τ, while the circles show the "hom" sets of all such arrows, the double shaft arrows are functions. The column at the base of the two triangles on the right shows the same three sets and two functions as objects and arrows in the category of sets.

Theorem 1.7.

$$\mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\} \cong \int_{C \in \mathbf{C}} \mathbf{D}\{\mathbf{g}‘C \leftarrow \mathbf{f}‘C\}$$

Proof. Let $S \in \mathbf{Set}$ be any fixed set, and α a set of functions indexed by objects $C \in \mathbf{C}$, so that $\forall C \centerdot \alpha_C : \mathbf{D}\{\mathbf{g}‘C \leftarrow \mathbf{f}‘C\} \longleftarrow S$ (in $\mathbf{Set}$) and let $t \in S$ be an arbitrary member of S. For any $A, B \in \mathbf{C}$, applying the left side of the wedge condition to t:

$$\begin{aligned}(\{\mathbf{g}`x \leftarrow \mathbf{f}`A\} \circ \alpha_A)`t &= (\{\mathbf{g}`x \leftarrow \mathbf{f}`A\}`(\alpha_A`t) \\ &= \mathbf{g}`x \circ (\alpha_A`t) \circ \mathbf{f}`A \\ &= \mathbf{g}`x \circ (\alpha_A`t)\end{aligned} \tag{1}$$

Applying the right side of the wedge condition:

$$(\{\mathbf{g}`B \leftarrow \mathbf{f}`x\} \circ \alpha_B)`t = \mathbf{g}`B \circ (\alpha_B`t) \circ \mathbf{f}`x = (\alpha_B`t) \circ \mathbf{f}`x \tag{2}$$

If α is a wedge, the left sides of (1) and (2) are equal, and so the right sides are too, that is, $(\{\mathbf{g}`x \leftarrow \mathbf{f}`A\} \circ \alpha_A)`t = (\{\mathbf{g}`B \leftarrow \mathbf{f}`x\} \circ \alpha_B)`t$ and so $\mathbf{g}`x \circ (\alpha_A`t) = (\alpha_B`t) \circ \mathbf{f}`x$. This equation shows that $(\alpha_{[]})`t$ is a natural transformation for all $t$, that is, if $\alpha$ is a wedge then $\forall t \in S \centerdot (\alpha_{[]})`t \in \mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\}$.

The notation may be a bit too elliptical. Note that $(\alpha_{[]})`t \neq \alpha_t$. Rather, $\alpha = \alpha_{[]}$ is an indexed family of functions and $(\alpha_{[]})`t$ is an indexed family of the results of applying each of them to t. More explictly,

$$\alpha_{[]} = (\mathbb{\lambda}\, t \in S \centerdot \mathbb{\lambda}\, C \in \mathbf{C} \centerdot \alpha_C`t) : \mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\} \longleftarrow S \text{ (in } \mathbf{Set}\text{)}$$

using "blackboard lambda", $\mathbb{\lambda}$, to match arguments with occurrences of openings.

A natural transformation $\tau \in \mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\}$ is a set of arrows $\tau_C : \mathbf{g}`C \longleftarrow \mathbf{f}`C$ (in $\mathbf{D}$) indexed by objects $C \in \mathbf{C}$. Given such an object, indexing by C becomes a function

$$\omega_C : \mathbf{D}\{\mathbf{g}`C \leftarrow \mathbf{f}`C\} \longleftarrow \mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\} \text{ (in } \mathbf{Set}\text{)}$$

defined by $\omega_C`\tau = \tau_C$.

Further, ω is a wedge, $\omega : \mathbf{D}\{\mathbf{g}`[] \leftarrow \mathbf{f}`[]\} \Leftarrow \mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\}$, because the wedge condition $\omega_x = \{\mathbf{g}`B \leftarrow \mathbf{f}`x\} \circ \omega_B = \{\mathbf{g}`x \leftarrow \mathbf{f}`A\} \circ \omega_A$ holds for each arrow $x : B \longleftarrow A$ (in $\mathbf{C}$).

To see this let $\tau \in \mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\}$ be an arbitrary natural transformation. By the equations (1&2) above $(\{\mathbf{g}`x \leftarrow \mathbf{f}`A\} \circ \omega_A)`\tau = (\mathbf{g}`x) \circ \tau_A = \tau_B \circ (\mathbf{f}`x) = (\{\mathbf{g}`B \leftarrow \mathbf{f}`x\} \circ \omega_B)`\tau$. Since τ was arbitrary, it must be that $\{\mathbf{g}`x \leftarrow \mathbf{f}`A\} \circ \omega_A = \{\mathbf{g}`B \leftarrow \mathbf{f}`x\} \circ \omega_B$, which is the wedge condition for ω.

As shown above, $\forall t \in S \centerdot \alpha_{[]}`t \in \mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\}$ and so $\alpha_{[]} : \mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\} \longleftarrow S$ (in $\mathbf{Set}$). In fact, this is an arrow in the category of wedges[7] $\alpha_{[]} : \omega \longleftarrow \alpha$ because $\forall C \centerdot \omega_C \circ \alpha_{[]} = \alpha_C$. This equation is easy to believe, because it looks like the definition of ω, but what it really amounts to is $(\mathbb{\lambda}\, t \centerdot (\mathbb{\lambda}\, C \centerdot \alpha)`C)`t \cong (\mathbb{\lambda}\, C \centerdot (\mathbb{\lambda}\, t \centerdot \alpha)`t)`C$. (???fix and explain that???)

The proof should be completed by showing that it is the *only* such arrow, and so ω is a terminal wedge and therefore isomorphic with the terminal wedge $\int_{C \in \mathbf{C}} \mathbf{D}\{\mathbf{g}`C \leftarrow \mathbf{f}`C\}$. □

Kelly [13, §2.1] defines the end $\int_A T(A, A)$ in an eniched category, and then uses the formula of the above theorem to define an enriched analog of the set of natual tramsformations and the functor category, thus swapping the roles of definition and theorem. Kelly [13, §2.2 (2.11)] writes E_A for what was been called ω_A here.

[7] $\mathrm{Wd}(P)$ defined by Loregian [17, 1.1.5, p.4].

1.3.5 Ends and Sheaves

Page 15: (1.27):

Given $Fe : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{D}$, Loregian defines the end of F as the equalizer

$$\int_C F(C,C) \cong eq \left(\prod_{C \in \mathbf{C}} F(C,C) \overset{F^*}{\underset{F_*}{\rightrightarrows}} \prod_{f:C \to C'} F(C,C') \right)$$

The product in the source, $\prod_{C \in \mathbf{C}} F(C,C)$, has no free variables and yet the product in the target, $\prod_{f:C \to C'} F(C,C')$ seems to be meant to depend upon C. The C in the source is not the C in the target.

There is a similar oddity in the received definition of a sheaf. for example, Johnstone [10, §0.2,p10] displays the following sheaf diagram:

> "Diagramatically, we can say that P is a sheaf iff the diagram
>
> $$P(U) \longrightarrow \prod_{\alpha} P(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} P(U_\alpha \cap U_\beta)$$
>
> is an equalizer, where the maps are induced by restictions in the obvious way."

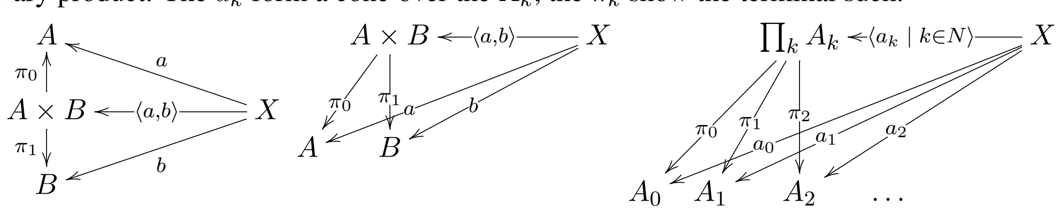
This use of the word "obvious" is obviously in the mathematician's sense where it means "I know what it is, but there is no standard notation for it and it would take too long to explain". Mac Lane and Moerdijk [19, §II.1(6),p67] give some more information on how to unscramble this. We need a name for the mediating arrow to a product which is constructed from an indexed family of arrows to its factors. In the case of two arrows in the family, say $a \colon A \longleftarrow X$ and $b \colon B \longleftarrow X$, we write $\langle a, b \rangle \colon A \times B \longleftarrow X$ for the unique arrow $m = \langle a, b \rangle$ such that $\pi_0 \circ m = a$ and $\pi_1 \circ m = b$.

Here a and b are just two arbitrary arrows, one targeted to each of the factors. They can be indexed by setting $A_0 = A$ And $A_1 = B$, $a_0 = a$, and $a_1 = b$, then $a_n \colon A_n \longleftarrow X$. Now

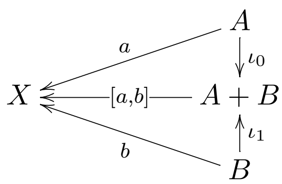
$$\langle a, b \rangle = \langle a_0, a_1 \rangle = \langle a_k \rangle_{k \in 2} = \langle a_k | k \in 2 \rangle \colon A \times B = A_0 \times A_1 = \textstyle\prod_{k \in 2} A_k \longleftarrow X$$

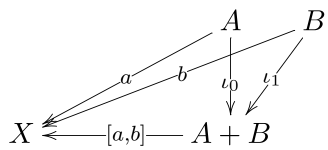
is the unique arrow such that $\forall n \in 2 \,.\, \pi_n \circ \langle a_k \rangle_{k \in 2} = a_n$ The index set 2 can be replaced by a larger index set.

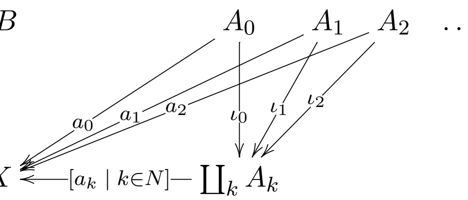
The drawings below show a binary product, the same folded in half on the center line, and an N-ary product. The a_k form a cone over the A_k, the π_k show the terminal such.



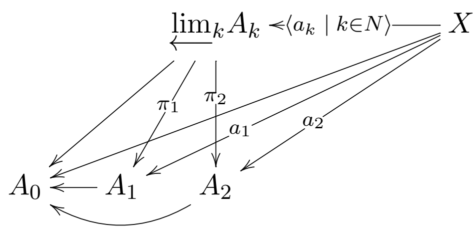
The next drawings show co-products. The a_k form a co-cone under the A_k, the ι_k show the initial (co-terminal) such.

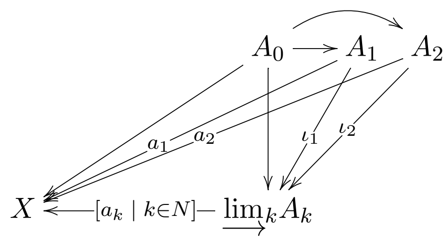






Drawings of more general limits and co-limits (inverse and direct limits) look much the same, but there are extra arrows between the various A_k. Because the diagram must still commute with the extra arrows added, some co/cones will be removed, but those that remain look the same as before.





One of the products in the sheaf diagram is indexed over α, the other over α, β. That seems to imply that α plays a role different from that of β, but that is not true. Both are bound variables; confusion can be reduced by choosing a different name for one of the bound variables.

$$\prod_{\alpha,\beta} P(U_\alpha \cap U_\beta) \leftleftarrows \prod_{\gamma} P(U_\gamma) \longleftarrow P(U)$$

Now $\prod_{\alpha,\beta} X_{\alpha,\beta} \cong \prod_\alpha \prod_\beta X_{\alpha,\beta} \cong \prod_\beta \prod_\alpha X_{\alpha,\beta}$ and the two arrows in the sheaf diagram are

$$\begin{aligned} \langle\langle \rho^{U_\alpha}_{U_\alpha \cap U_\beta} \circ \pi_\alpha \mid \alpha\rangle \mid \beta\rangle &: \textstyle\prod_\beta \prod_\alpha P(U_\alpha \cap U_\beta) \longleftarrow \prod_\gamma P(U_\gamma) \quad \text{and} \\ \langle\langle \rho^{U_\beta}_{U_\alpha \cap U_\beta} \circ \pi_\beta \mid \beta\rangle \mid \alpha\rangle &: \textstyle\prod_\alpha \prod_\beta P(U_\alpha \cap U_\beta) \longleftarrow \prod_\gamma P(U_\gamma) \end{aligned} \tag{3}$$

Where $\rho^U_V = P(U \supseteq V) : P(V) \longleftarrow P(U)$ is the restriction map $P(U \supseteq V) = P(i)$ where $i : U \supseteq V$ is the inclusion map $i`v = v \in U$ if $v \in V$. Remember that P is contravariant, and so maps inclusion to restriction.

In a similar way, given $F : \mathbf{D} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$, the product in the target of the diagram (1.26)

$$\prod_{f:C' \leftarrow C} F(C, C') \overset{F^*}{\underset{F_*}{\leftleftarrows}} \prod_{X \in \mathbf{C}} F(X, X) \overset{=}{\longleftarrow} \int_C F(C, C)$$

can be written as

$$\prod_{f:C' \leftarrow C} F(C, C') \cong \prod_{C \in \mathbf{C}} \prod_{C' \in \mathbf{C}} \prod_{f:C' \leftarrow C} F(C, C')$$

Loregian [17, p13.9] says:

> the arrows F^*, F_* are easily obtained from the arrows whose $(f; C, C')$-components are (respectively) $F(f, C')$ and $F(C, f)$.

Remember that $F(f, C') \mathbf{:} F(C, C') \longleftarrow F(C', C')$ and $F(C, f) \mathbf{:} F(C, C') \longleftarrow F(C, C)$, and then define $F^* = \prod_{C' \in \mathbf{C}} (F^*)_{C'}$ and $F_* = \prod_{C \in \mathbf{C}} (F_*)_C$ where

$$(F^*)_{C'} = \langle \langle F(f, C') \mid f : C' \leftarrow C \rangle \circ \pi_{C'} \mid C' \in \mathbf{C} \rangle : \prod_{C' \in \mathbf{C}} \prod_{f : C' \leftarrow C} F(C, C') \longleftarrow \prod_{X \in \mathbf{C}} F(X, X)$$

$$(F_*)_C = \langle \langle F(C, f) \mid f : C' \leftarrow C \rangle \circ \pi_C \mid C \in \mathbf{C} \rangle : \prod_{C \in \mathbf{C}} \prod_{f : C' \leftarrow C} F(C, C') \longleftarrow \prod_{X \in \mathbf{C}} F(X, X)$$

1.3.6 Representation and Universality

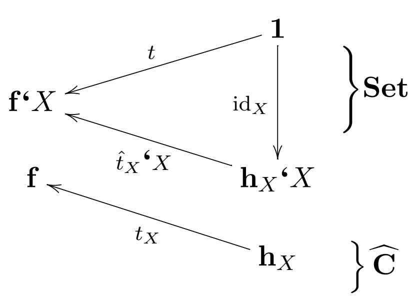
Page 266.5:

Loregian (at the place cited above)

> "The identity arrow id_X can be thought as the *universal element* witnessing the representability of a functor"

CWM [18, p57] defines universal element. He begins "If D is a category and $H : D \to \mathbf{Set}$ a functor …". So what is the category and what the functor? Also, what is a witness for representability?

Also CWM [p61] says a representation of of a functor $\mathbf{k} \mathbf{:} \mathbf{Set} \longleftarrow \mathbf{D}$ is an object $R \in \mathbf{D}$ together with a natural isomorphism ψ $\mathbf{::} \mathbf{D}\{[] \leftarrow R\} \Longleftrightarrow \mathbf{k} \mathbf{:} \mathbf{Set} \longleftarrow \mathbf{D}$.



An element of a set is equivalent to an arrow to the set from the terminal object in the category of sets. So a universal element of a $\mathbf{Set}$-valued functor is is a universal arrow to the functor from $\mathbf{1}$.

In the diagram to the left, the triangle is in the category of sets while the lower arrow is in the category $\widehat{\mathbf{C}} = \mathbf{Set}^\frown \mathbf{C}^{op}$ of presheaves on $\mathbf{C}$. Furthermore, $\mathbf{f} \mathbf{:} \mathbf{Set} \longleftarrow \mathbf{C}^{op}$ is an arbitrary presheaf, $X \in \mathbf{C}$ is an object of $\mathbf{C}$, and $\mathbf{h}_X = \mathbf{C}\{X \leftarrow []\}$ is the presheaf associated with X by the Yoneda lemma..

Thus id_X is a universal element of the functor $[]$‘$X \mathbf{:} \mathbf{Set} \longleftarrow \mathbf{Set}^\frown \mathbf{C}^{op}$ when applied to $\mathbf{h}_X$. This entails that there is an isomorphism, call it ψ, such that

$$\psi \mathbf{::} \widehat{\mathbf{C}}\{[] \leftarrow \mathbf{h}_X\} \Longleftrightarrow [\,]‘X \ \mathbf{:} \mathbf{Set} \longleftarrow \widehat{\mathbf{C}}$$

In particular, for any $\mathbf{f} \in \widehat{\mathbf{C}}$, there is a bijection (isomorphism in the category of sets)

$$\psi_{\mathbf{f}} : \widehat{\mathbf{C}}\{\mathbf{f} \leftarrow \mathbf{h}_X\} \cong \mathbf{f}‘X (\in \mathbf{Set})$$

when it is proven that this bijection is natural in $\mathbf{f}$, just define "witnessing the representability" as the relation between the identity id_X and the functor $[\,]$‘X (apply to X) that makes it all work. Then Loregian's words can be interpreted as truth.

1.3.7 Rants

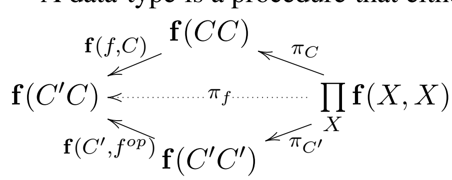
(Uncalled for):

We have covariant and contravariant functors. That suggests that the dual of a colimit should be a contralimit. And the cosine of x is the contrasine of $x + \pi/2$. That way lies madness; we can't re-make the world, or even re-name all of it. Still, I have read the phrase "complete and co-complete" often enough to want a way to say both. Bicomplete looks like it is for bicategories. "X, co-X, co/X" works in writing, but it's hard to pronounce. "Ambi-" means "both" in some dead language, cf. ambidextrous. ambiguous. "Complete, co-complete, ambi-complete"; maybe pronounce it as am-bi-complete.

Are we are doomed by the extended Yoneda Lemma? A category embeds covariantly into a contravariant functor category, and contravariantly into a covariant functor category, and furthermore, no matter how you bang on it to try to straighten it out, it will be contra-something.

Stokes' theorem arises from glueing simplexes (simple polytopes) together to make a differentiable manifold. Is it possible to generalize it to cover glueing simple opetopes to make a computational "manifold"? See Baez and Dolan [1] and Cheng [4].

A data-type is a procedure that either does nothing or fails.



If this is commutative then $\pi : \mathbf{f} \Leftarrow \prod_{X\in\mathbf{C}} \mathbf{f}(X,X)$ is a wedge. (The wedge of pi.) This is a property of $\mathbf{f}$, a property enjoyed by $\mathbf{hom}$ when $\mathbf{D} = \mathbf{Set}$. Can the whole theory of twisted arrows be generalized to enriched categories?

A $\mathcal{V}$-enhanced category is a category $\mathbf{C}$ together with a a functor $\mathcal{V}\text{-}\mathbf{hom} : \mathcal{V} \longleftarrow \mathbf{C} \times \mathbf{C}^{op}$, which satisfies the following axioms:

Unital $\mathcal{V}\text{-}\mathbf{hom}`(f, C) = \mathcal{V}\text{-}\mathbf{hom}`(C', f^{op})$

Associative $\mathcal{V}\text{-}\mathbf{hom}`(g \circ a, f^{op}) = \mathcal{V}\text{-}\mathbf{hom}`(g, (a \circ f)^{op})$

2 Revised old notes

These are part of some old notes on *A Generalization of the Functorial Calculus* by Eilenberg and Kelly [7] that have been revised as I read *Coend Calculus*.

2.1 Transformations: natural, co-wedge, wedge

The phrase "extraordinary natural transformations" is used in the original paper by Eilenberg and Kelly [7], which introduced the concept, but seems not to have become standard. Mac Lane [18, CWM IX§4] mentions the words "dinatural", "extranatural", "supernatural", and "wedge"[8], for various special cases. We have no use for unnatural transformations, so let's call them all simply "transformations", reserving the the adjective "natural" for the ordinary case of a natural transformation between parallel functors as defined in most any introduction to category theory (including CWM [18, p.16,§I.4]).

Natural, wedge, and co-wedge transformations, are exhibited, but not separately named, by Eilenberg and Kelly. They say "the family $\alpha = \{\alpha(ABC)\}$ is *natural* if" the following definition applies:

Definition 2.1. *Suppose given two functors*

$$\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B} \qquad \textit{and} \qquad \mathbf{s} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C}$$

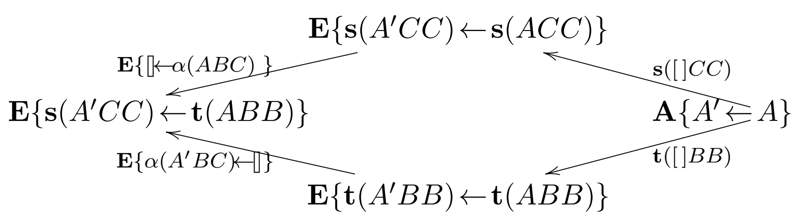
and a family of arrows in $\mathbf{E}$ *indexed by objects* $A \in \mathbf{A}$, $B \in \mathbf{B}$, $C \in \mathbf{C}$

$$\Big(\alpha(ABC) : \mathbf{s}(ACC) \longleftarrow \mathbf{t}(ABB)\Big)_{A\in\mathbf{A}, B\in\mathbf{B}, C\in\mathbf{C}}$$

This is a (natural or) extranatural family or is a transformation if the following three diagrams commute for all $A, A' \in \mathbf{A}$, *and* $B, B' \in \mathbf{B}$, *and* $C, C' \in \mathbf{C}$.

These first three diagrams are in the category of sets. In these diagrams, to save space, I use "[]" to stand for an identity arrow. Context will suffice to determine the object on which it is an identity.

The symbol ↺ marks a contravariant functor.



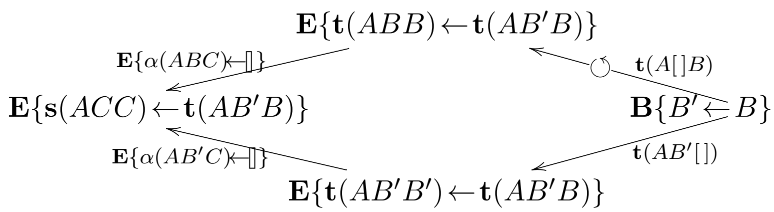
(1)

Natural transformation

$$\mathbf{E}\{[\,] \leftarrow \alpha(ABC)\} \circ \mathbf{s}([\,]CC) = \mathbf{E}\{\alpha(A'BC) \leftarrow [\,]\} \circ \mathbf{t}([\,]BB)$$

In this diagram the "[]" in the target of the hom set function labeling the arrow in the upper left denotes the identity on $\mathbf{s}(A'CC)$, while that in the source of the lower left hom function denotes the identity on $\mathbf{t}(ABB)$. This can be seen by looking at the source and target of the labeled arrow.

[8] I once said: "CWM calls them both 'wedges'. There is no 'cowedge' perhaps for fear of cows with edges.", but that comment seems to have been made obsolete.

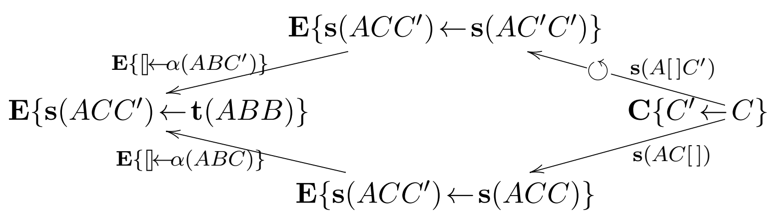


(2)

Co-wedge transformation

$$\mathbf{E}\{\alpha(ABC)\leftarrow[]\}\circ\mathbf{t}(A[]B)=\mathbf{E}\{\alpha(AB'C)\leftarrow[]\}\circ\mathbf{t}(AB'[])$$

Here both occurrences of "[]" denote the identity on $\mathbf{t}(AB'B)$.



(3)

Wedge transformation

$$\mathbf{E}\{[]\leftarrow\alpha(ABC')\}\circ\mathbf{s}(A[]C')=\mathbf{E}\{[]\leftarrow\alpha(ABC)\}\circ\mathbf{s}(AC[])$$

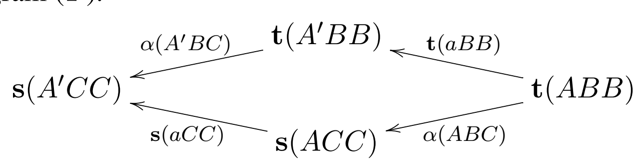
Here "[]" denotes the identity on $\mathbf{s}(ACC')$.

The assertion that α is a transformation to $\mathbf{s}$ from $\mathbf{t}$ can be symbolized as $\alpha:\mathbf{s}\Leftarrow\mathbf{t}$. This symbolic form is taken from Loregian [17] who may have taken it from Yoneda. An ordinary arrow would not be appropriate, because there is no category of transformations. We already have a symbolic notation for a natural transformation.

We need a short way to annotate the symbolic form with information which can be used to calculate whether $\beta:\mathbf{r}\Leftarrow\mathbf{s}$ can be composed with $\alpha:\mathbf{s}\Leftarrow\mathbf{t}$ to get $\beta\circ\alpha:\mathbf{r}\Leftarrow\mathbf{t}$. In the case of ordinary, or even enriched, categories, the identity of the source of β with the target of α is sufficient. We presumably also need the Kelly-Mac Lane graph that connects the two transformations, but how is it to be written as text?

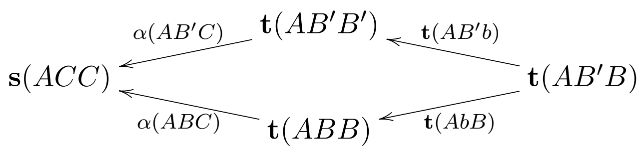
The next three diagrams are in $\mathbf{E}$. They follow from the above correponding diagrams.

For example, to say that diagram (1) commutes is to say that for all $a\in\mathbf{A}\{A'\leftarrow A\}$ the two members of the set $\mathbf{E}\{\mathbf{s}(A'CC)\leftarrow\mathbf{t}(ABB)\}$ given by $\mathbf{E}\{[]\leftarrow\alpha(ABC)\}`\mathbf{s}(aCC)=\mathbf{s}(aCC)\circ\alpha(ABC)$ and by $\mathbf{E}\{\alpha(A'BC)\leftarrow[]\}`\mathbf{t}(aBB)=\alpha(A'BC)\circ\mathbf{t}(aBB)$ are equal. This is shown in diagram (1′).



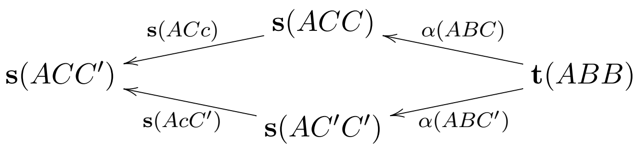
(1′)

Similarly, diagram (2) commutes if for all $b\in\mathbf{B}\{B'\leftarrow B\}$ the two members of the set of arrows $\mathbf{E}\{\mathbf{s}(ACC)\leftarrow\mathbf{t}(AB'B)\}$ given by $\mathbf{E}\{\alpha(ABC)\leftarrow[]\}`\mathbf{t}(AbC)=\alpha(ABC)\circ\mathbf{t}(AbC)$ and by $\mathbf{E}\{\alpha(AB'C)\leftarrow[]\}`\mathbf{t}(AB'b)=\alpha(A'BC)\circ\mathbf{t}(AB'b)$ are equal. This shown in diagram (2′).



(2′)

Finally, (3) commutes if $\forall c \in \mathbf{C}\{C' \leftarrow C\}$ the members of $\mathbf{E}\{\mathbf{s}(ACC') \leftarrow \mathbf{t}(ABB)\}$ given by $\mathbf{E}\{[] \leftarrow \alpha(ABC)\}`\mathbf{s}(ACc) = \mathbf{s}(ACc) \circ \alpha(ABC)$ and by $\mathbf{E}\{[] \leftarrow \alpha(ABC')\}`\mathbf{s}(AcC') = \mathbf{s}(AcC') \circ \alpha(ABC')$ are equal. This shown in diagram (3′).



(3′)

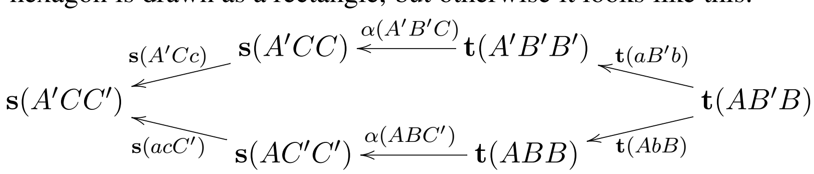
Perhaps it is worthwhile to go a little more carefully through one of those computations. Diagram (1) manifestly says that $\mathbf{E}\{[] \leftarrow \alpha(ABC)\} \circ \mathbf{s}([\,]CC) = \mathbf{E}\{\alpha(A'BC) \leftarrow []\} \circ \mathbf{t}([\,]BB)$; this means that $(\mathbf{E}\{[] \leftarrow \alpha(ABC)\} \circ \mathbf{s}([\,]CC))`a = (\mathbf{E}\{\alpha(A'BC) \leftarrow []\} \circ \mathbf{t}([\,]BB))`a$ for all $a : A' \longleftarrow A (\in \mathbf{A})$. Further, by definition of $\circ$ in $\mathbf{Set}$, we have $(g \circ f)`a = g`(f`a)$ so since $\mathbf{s}([\,]CC))`a = \mathbf{s}(aCC))$ by the action of the **hom** functor on arrows $\mathbf{E}\{[] \leftarrow \alpha(ABC)\}`\mathbf{s}(aCC) = [] \circ \mathbf{s}(aCC) \circ \alpha(ABC) = \mathbf{s}(aCC) \circ \alpha(ABC)$. Other equalities are similar.

Diagram (1′) shows a natural transformation parameterized by B and C. Diagram (2′) shows a co-wedge transformation parameterized by A and C. Diagram (3′) shows a wedge transformation parameterized by A and B.

If in (3′) the functor $\mathbf{t}$ ignores its arguments, A and B, and yields a constant object, say $E \in \mathbf{E}$, we write $\alpha : \mathbf{s} \Leftarrow E$ and call α a wedge. Similarly, if in (2′) the functor $\mathbf{s}$ is constant, we write $\alpha : E \Leftarrow \mathbf{t}$ and call α a co-wedge.

Eilenberg and Kelly [7, (4),p368] display something which I call the "Hexagon of Transformation". They don't call it anything except "a single commuting diagram" into which (1), (2), (3) can be "condensed". I take the word "condensed" as a claim that the single diagram can be substituted for the original three to give an equivalent definition.

Their hexagon is drawn as a rectangle, but otherwise it looks like this:



(4)

If it is not obvious how the three diagrams (1'), (2'), (3') condense into the one hexagon, stare at Figure 6 and note that the three narrow parallelograms (forming a backward 'Z' shape in the middle) are instances of (1'), while the scalene quadrilaterals on the right and left are (2') and (3'), respectively.

Because $\mathbf{t}(aB'b) = \mathbf{t}(aB'B') \circ \mathbf{t}(AB'b)$ and $\mathbf{s}(acC') = \mathbf{s}(A'cC') \circ \mathbf{s}(AC'C')$, the perimeters of (4) and Figure 6 are the same. Figure 7 shows exactly the same objects and arrows rearranged to make it easier to see how the wedge, three squares, and cowedge fit together.

In the other direction, the three diagrams are obtained from the Hexagon by setting
(1') $b = B' = B$ and $c = C' = C$, so $\mathbf{t}(ABB) = \mathbf{t}(AB'B)$ and $\mathbf{s}(A'CC) = \mathbf{s}(A'CC')$
(2') $a = A' = A$ and $c = C' = C$, so $\mathbf{s}(ACC) = \mathbf{s}(A'CC) = \mathbf{s}(AC'C) = \mathbf{s}(A'CC')$

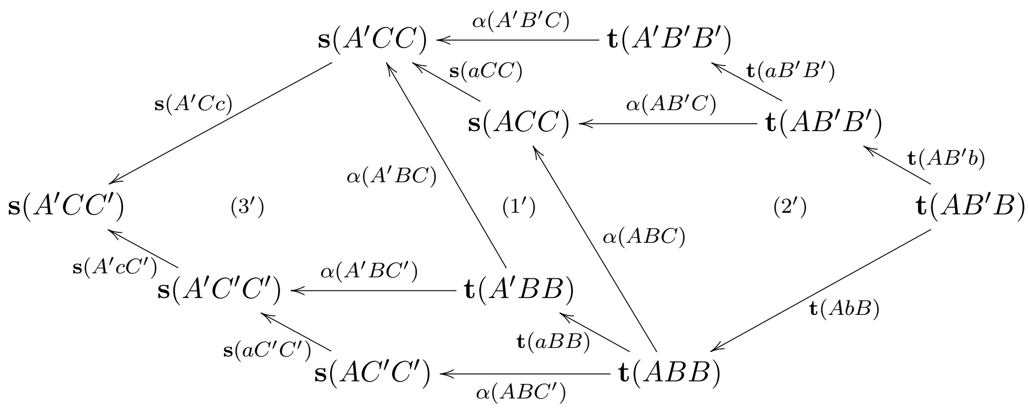


Figure 6: Making the Hexagon of Transformation

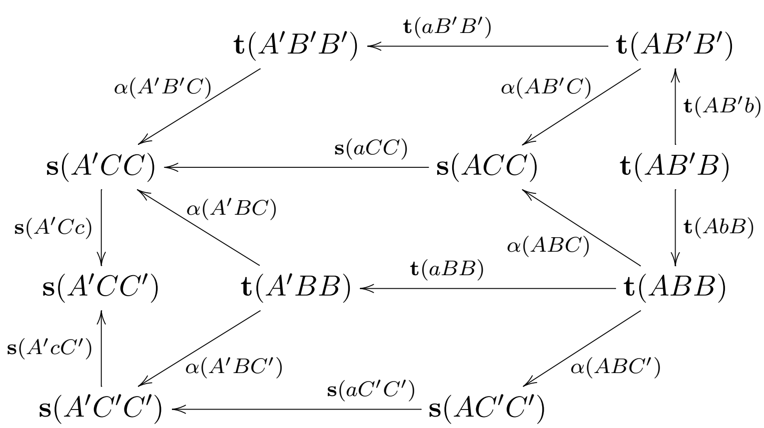


Figure 7: The rectified hexagon

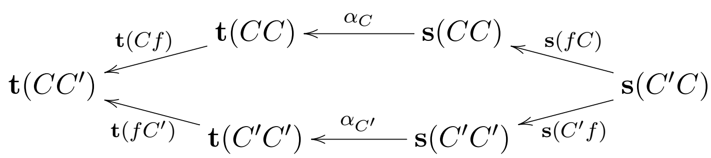
(3') $a = A' = A$ and $b = B' = B$, so $\mathbf{t}(ABB) = \mathbf{t}(AB'B) = \mathbf{t}(A'B'B') = \mathbf{t}(A'B'B')$ respectively.

Given functors $\mathbf{s} : \mathbf{E} \leftarrow \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C}$ and $\mathbf{t} : \mathbf{E} \leftarrow \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B}$, using either the three squares or the hexagon, Eilenberg and Kelly define a transformation $\alpha : \mathbf{s} \Leftarrow \mathbf{t}$, to be an indexed family of arrows $\alpha(ABC) : \mathbf{s}(ACC) \leftarrow \mathbf{t}(ABB)$ that make the diagram(s) commute. (Actually, E&K does not call this a "transformation", but rather say that the family is "natural" in an extended sense.)

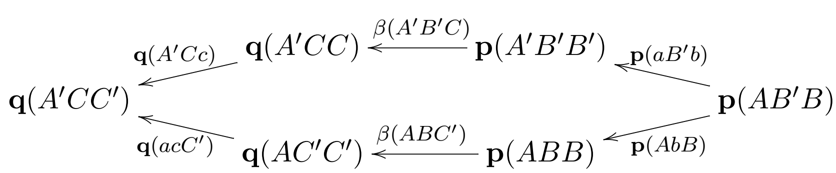
2.2 Dinatural and Extranatural Transformations

Saunder Mac Lane, in his famous book *Categories for the Working Mathematician*, defines a dinatural transformation. Diagonal naturality, as described by Mac Lane [18, CWM §IX.4], is a special case of extranaturality, as defined by Eilenberg and Kelly [7].
NB: CWM uses T for the target and S for the source of the dinatural transformation. This reasonable convention is the opposite of that originally used by Eilenberg and Kelly [7].

Given parallel functors $\mathbf{s}, \mathbf{t} : \mathbf{E} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$, a dinatural transformation $\alpha : \mathbf{t} \overset{\bullet\bullet}{\longleftarrow} \mathbf{s}$ in the sense of Mac Lane [18, CWM §IX.4] is a family of arrows $\alpha_C : \mathbf{t}(C, C) \longleftarrow \mathbf{s}(C, C)$ indexed by objects $C \in \mathbf{C}$ such that for every arrow $f : C' \longleftarrow C (\in \mathbf{C})$ the hexagon of dinaturality commutes. Mac Lane's hexagon of dinaturality looks like this:



To avoid confusion, let's rename the functors in the E&K definition of a transformation[9] so that $\beta : \mathbf{q} \Leftarrow \mathbf{p}$ where $\mathbf{q} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C}$ and $\mathbf{p} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B}$:



Let $\mathbf{I}$ be the category with one object and one arrow (the identity), both called $*$.

The following theorem can be summarized as saying that a dinatural transformation is a special case of (extranaural) transformation. Specifically, it is a transformation where the categories $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, have a special form, and the indexing of the transformational family of arrows is also specialized in a way that is justified by the special form of those three categories.

Theorem 2.2. *Suppose we are given functors* $\mathbf{q} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{C}^{op} \times \mathbf{C}$*, and* $\mathbf{p} : \mathbf{E} \longleftarrow \mathbf{A} \times \mathbf{B}^{op} \times \mathbf{B}$ *where* $\mathbf{B} = \mathbf{C}^{op}$ *and* $\mathbf{A} = \mathbf{I}$*. If* $\beta : \mathbf{q} \Leftarrow \mathbf{p}$ *is a transformation then the family* $\alpha_C = \beta(*CC)$ *is dinatural* $\alpha : \mathbf{s} \overset{\bullet\bullet}{\longleftarrow} \mathbf{t}$ *where* $\mathbf{s}, \mathbf{t} : \mathbf{E} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$*, and* $\mathbf{t}(xy) = \mathbf{q}(*xy)$*, and* $\mathbf{s}(yx) = \mathbf{p}(*xy)$*.*

[9] By happy coincidence the object names exactly match those of Loregian's diagram (1.16) [17, p.7].

Proof. Assume given $\mathbf{q}$ and $\mathbf{p}$ as above and that $\beta : \mathbf{q} \Leftarrow \mathbf{p}$ is a transformation. Define $\mathbf{s}$, $\mathbf{t}$, and α as above; Note that the arguments of $\mathbf{s}$ are swapped while those of $\mathbf{t}$ are not.

It must be shown that the dinaturality hexagon commutes for every $f : C' \longleftarrow C$. For any such f, let $b = f$ and $c = f$. Since $\mathbf{B} = \mathbf{C}^{op}$, $\mathbf{B}$ and $\mathbf{C}$ have the same objects, so set $B = C'$ and $B' = C$, so that, $b = f : C \longleftarrow C'(\in \mathbf{C}^{op})$ or equivalently $b : B' \longleftarrow B(\in \mathbf{B})$ and $c : C' \longleftarrow C(\in \mathbf{C})$.

Now $\mathbf{q} : \mathbf{E} \longleftarrow \mathbf{I} \times \mathbf{C}^{op} \times \mathbf{C}$ and $\mathbf{p} : \mathbf{E} \longleftarrow \mathbf{I} \times \mathbf{B}^{op} \times \mathbf{B}$ equivalently $\mathbf{p} : \mathbf{E} \longleftarrow \mathbf{I} \times \mathbf{C} \times \mathbf{C}^{op}$ and so $\mathbf{t} : \mathbf{E} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$ and $\mathbf{s} : \mathbf{E} \longleftarrow \mathbf{C}^{op} \times \mathbf{C}$.

Now we must show that the dinaturality hexagon commutes, that is

$$\begin{aligned} \mathbf{t}(Cf) \circ \alpha_C \circ \mathbf{s}(fC) &= \mathbf{t}(fC') \circ \alpha_{C'} \circ \mathbf{s}(C'f) && \text{which is} \\ \mathbf{q}(*Cf) \circ \beta(*CC) \circ \mathbf{p}(*Cf) &= \mathbf{q}(*fC') \circ \beta(*C'C') \circ \mathbf{p}(*fC') && \text{which is} \\ \mathbf{q}(*Cc) \circ \beta(*B'C) \circ \mathbf{p}(*B'b) &= \mathbf{q}(*cC') \circ \beta(*BC') \circ \mathbf{p}(*bB) \end{aligned}$$

which is exactly the hexagon of transformation. □

Well, maybe not exactly. According to CWM [18, §II.1.2] the category $\mathbf{C}^{op}$ has arrows f^{op}. I wanted to write $b = f^{op}$ above but have not yet made that work. Maybe I'm not smart enough, but maybe Mac Lane's hexagon does not follow Mac Lane's convention. In particular, "f" occurs four times, twice where an arrow of $\mathbf{C}^{op}$ is expected and twice where an arrow of $\mathbf{C}$ is expected. Is $f = f^{op}$ after all, or is there an application of a bijection which is not explicitly written?

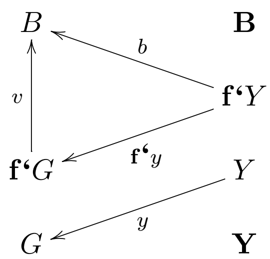


Figure 8: Co-free object G, together with Universal arrow from $\mathbf{f}, G$ v to B

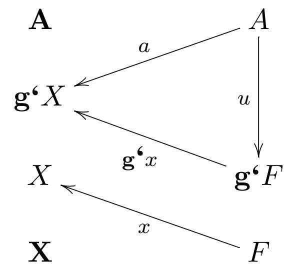


Figure 9: Free object F, together with Universal arrow u to $\mathbf{g}, F$ from A

3 Category Theory

3.1 Ordinary (Set-Enriched) Categories

In my notes this is an introduction to category theory, but it is omitted in this version.

3.1.1 Universal Arrows and Free objects

Look at figure 9. Given a functor $\mathbf{g}\,\mathbf{:}\,\mathbf{A} \longleftarrow \mathbf{X}$, and an object $F \in \mathbf{X}$, we say that F is a free object iff there is an arrow $u\,\mathbf{:}\,\mathbf{g}\text{‘}F \longleftarrow A$ $(\text{in}\mathbf{A})$, called a universal arrow to $(\mathbf{g}, F)$ from A, such that for any object $X \in \mathbf{X}$ and arrow $a\,\mathbf{:}\,\mathbf{g}\text{‘}X \longleftarrow A(\text{in}\mathbf{A})$ there is a unique $x\,\mathbf{:}\,X \longleftarrow F(\text{in}\mathbf{X})$ such that $a = \mathbf{g}\text{‘}x \circ u$, in other words, every arrow from A to an object in the image of $\mathbf{g}$ factors uniquely into an arrow in the image of $\mathbf{g}$ following the universal arrow.

The motivating example is where $\mathbf{A}$ is the category of sets and $\mathbf{X}$ is some variety of algebras, such as monoids or groups. In that case F is the free algebra on the set A of generators and $\mathbf{g}$ is the forgetful (grounding) functor that maps an algebraic structure to its underlying set of elements (forgetting the structure). The universal arrow is the injection of the set of generators into itself seen as a subset of the elements of the free algebra.

Note that although the target of u is $\mathbf{g}\text{‘}F$ we must say that it is universal to $(\mathbf{g}, F)$ because it is possible that $\mathbf{g}\text{‘}F = \mathbf{g}\text{‘}F_2$ but u is *not* universal to $(\mathbf{g}, F_2)$. Think of F as a free group and F_2 as another group with the same cardinality (it might as well be the same elements), but with a completely different rule of multiplication.

There is a dual construction. Look at figure 8.

Given a functor $\mathbf{f}\,\mathbf{:}\,\mathbf{B} \longleftarrow \mathbf{Y}$ and an object $G \in \mathbf{Y}$ we say that G is a co-free object iff there is an arrow $v\,\mathbf{:}\,B \longleftarrow \mathbf{f}\text{‘}G$ $(\text{in } \mathbf{B})$, called a universal arrow from $(\mathbf{f}, G)$ to B, such that for any object $Y \in \mathbf{Y}$ and arrow $b\,\mathbf{:}\,B \longleftarrow \mathbf{f}\text{‘}Y$ $(\text{in } \mathbf{B})$ there is a unique $y\,\mathbf{:}\,G \longleftarrow Y$ $(\text{in } \mathbf{Y})$ such that $b = v \circ \mathbf{f}\text{‘}y$, in other words, every arrow to B from an object in the image of $\mathbf{f}$ factors uniquely into the universal arrow following an arrow in the image of $\mathbf{f}$.

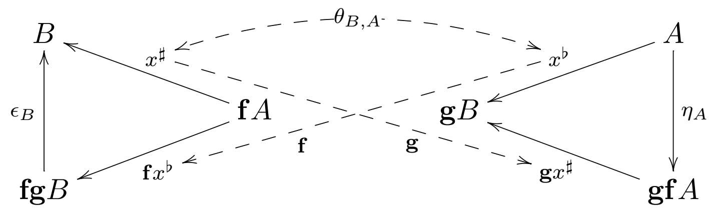


Figure 10: Unit and co-unit of adjunction

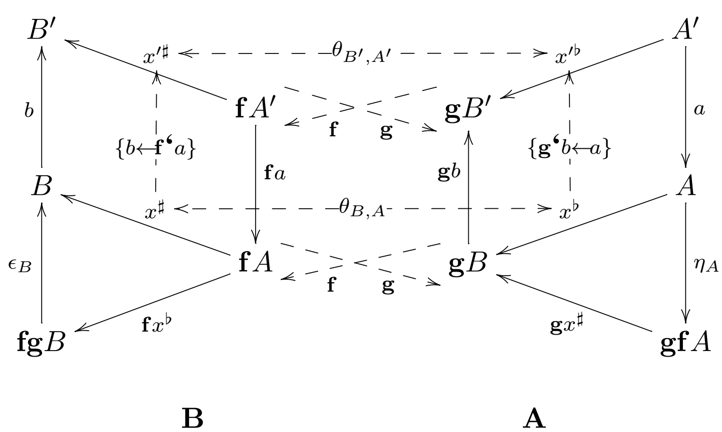


Figure 11: Adjunction of $\mathbf{f}$ and $\mathbf{g}$

3.1.2 Adjunction

Definition 3.1. *Given functors* $\mathbf{f} : \mathbf{B} \longleftarrow \mathbf{A}$ *and* $\mathbf{g} : \mathbf{A} \longleftarrow \mathbf{B}$ *we say that* $\mathbf{f}$ *is a* left adjoint *of* $\mathbf{g}$*,* $\mathbf{g}$ *is a* right adjoint *of* $\mathbf{f}$*, and* θ *is an* adjunction *to* $\mathbf{f}$ *from* $\mathbf{g}$*, written* $\theta :: \mathbf{f} \dashv \mathbf{g}$*, if*

$$\theta_{B,A} : \mathbf{B}\{B \leftarrow \mathbf{f}`A\} \longleftrightarrow \mathbf{A}\{\mathbf{g}`B \leftarrow A\}$$

is a bijection natural in A *and* B*.*

This is shown in figure 10. In that figure, first let $B = \mathbf{f}`A$ and let $x^\sharp = [_{\mathbf{f}`A}]$ be the identity, (so is $\mathbf{g}`x^\sharp$), so that $\eta_A = x^\flat$, and next that $x^\flat$ is the identity, and so is $\mathbf{f}`x^\flat$, so that $\epsilon_B = x^\sharp$,

The unit and co-unit are each natural transformations:

$$\epsilon :: [\,] \Longleftarrow \mathbf{f} \circ \mathbf{g} : A \longleftarrow A$$

$$\eta :: \mathbf{g} \circ \mathbf{f} \Longleftarrow [\,] : B \longleftarrow B$$

.

3.1.3 Kan Extensions and Lifts

The right Kan extension of a functor $\mathbf{b} : \mathbf{X} \longleftarrow \mathbf{B}$ along a functor $\mathbf{h} : \mathbf{A} \longleftarrow \mathbf{B}$ is just a universal arrow to $\mathbf{b}$ from $\mathbf{X}^{\mathbf{h}}$, $\mathbf{a}$. in the functor category $\mathbf{X}^{\mathbf{B}}$. See figure 13.

Note that in figure 12, which shows the left Kan extension, $\mathbf{k}$ points in the opposite direction, $\mathbf{k} : \mathbf{B} \longleftarrow \mathbf{A}$. The extension is now a universal arrow in the functor category $\mathbf{X}^{\mathbf{A}}$ Note $\mathbf{X}^{\mathbf{h}} = [\,] \circ \mathbf{h} : \mathbf{X}^{\mathbf{B}} \longleftarrow \mathbf{X}^{\mathbf{A}}$.

Suppose $[_X] \circ \mathbf{t} \circ [_B] \circ \pi \circ [_{(A \downarrow \mathbf{k})}]$ for each $A \in \mathbf{A}$ has a limit in $\mathbf{X}$. Let λ be the limiting cone and $\mathbf{f}`A = \varprojlim \left(\mathbf{t} \circ \pi \circ [_{(A \downarrow \mathbf{k})}]\right)$.

Each $a : A' \longleftarrow A$ induces a unique arrow $\mathbf{f}`a : \varprojlim \mathbf{t} \circ \pi' \longleftarrow \varprojlim \mathbf{t} \circ \pi$ commuting with the limiting cones. This defines a functor $\mathbf{f} : \mathbf{X} \longleftarrow \mathbf{A}$ and for each $b \in \mathbf{B}$ the components $\lambda_{[\mathbf{k}`b]} = \epsilon_n$ define a natural transformation $\epsilon : \mathbf{t} \longleftarrow \mathbf{f} \circ \mathbf{k}$ and $\mathbf{f}, \epsilon$ is a right Kan extension.

If $\mathbf{X} = \mathbf{B}$ then $\mathrm{Ran}_{\mathbf{k}}[_{\mathbf{B}}] \dashv \mathbf{k}$ and $\mathrm{Ran}_{\mathbf{k}}\mathbf{b} = \mathbf{b} \circ \mathrm{Ran}_{\mathbf{k}}[_{\mathbf{B}}]$ This makes it look as if left and right are reversed in the naming of Kan extensions.

Every functor $\mathbf{t} : \mathbf{X} \longleftarrow \mathbf{B}$ has a right Kan extension along the functor $\mathbf{k} : \mathbf{A} \longleftarrow \mathbf{B}$ if and only if $\mathbf{X}^{\mathbf{k}} \dashv \mathrm{Ran}_{\mathbf{k}}$. more fully $\mathbf{X}^{\mathbf{k}} : \mathbf{X}^{\mathbf{B}} \underset{\longrightarrow}{\overset{\longleftarrow}{\dashv}} \mathbf{X}^{\mathbf{A}} : \mathrm{Ran}_{\mathbf{k}}$. This makes it look as if left and right are correct.

The definition of Kan extension is taken mostly from Mac Lane [18], while Loregian [17, Rem.2.1.3, p.32] is the source of *everything* about Kan lifts. "Everything is a Kan extension", but I had never heard of a lift. Is this due to a theorem of category theory, or a quirk of psychological history?

3.1.3.1 Kan Extensions – Left and Right Given a functor $\mathbf{k} : \mathbf{B} \longleftarrow \mathbf{A}$, another functor $\mathbf{X}^{\mathbf{k}} : \mathbf{X}^{\mathbf{A}} \longleftarrow \mathbf{X}^{\mathbf{B}}$ between functor categories can be defined by $\mathbf{X}^{\mathbf{k}}`\mathbf{t} = \mathbf{T} \circ \mathbf{k} : \mathbf{X} \longleftarrow \mathbf{B}$. for any $\mathbf{t} \in \mathbf{X}^{\mathbf{B}}$ (that is, $\mathbf{t} : \mathbf{X} \longleftarrow \mathbf{B}$).

Briefly, $\mathbf{X}^{\mathbf{k}} = \lambda \mathbf{t} \,.\, \mathbf{t} \circ \mathbf{k} = [\,] \circ \mathbf{k} : \mathbf{X}^{\mathbf{A}} \longleftarrow \mathbf{X}^{\mathbf{B}}$.

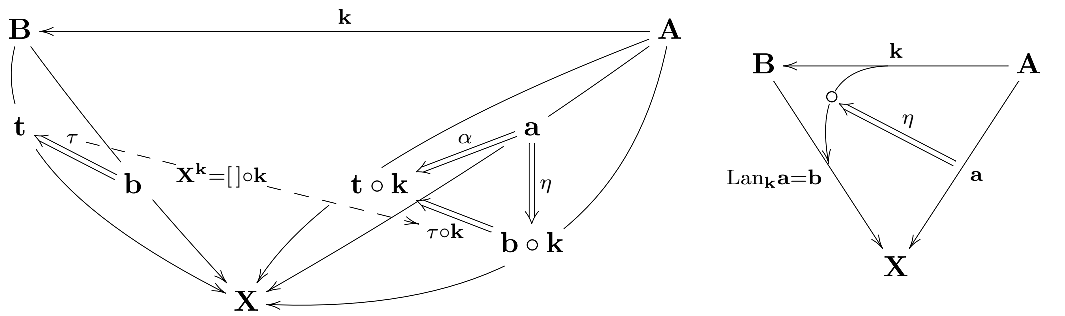


Figure 12: $\mathbf{b} : \mathbf{X} \longleftarrow \mathbf{B}$ is the left Kan extension of $\mathbf{a} : \mathbf{X} \longleftarrow \mathbf{B}$ along $\mathbf{k} : \mathbf{B} \longleftarrow \mathbf{A}$ (that is $\mathbf{b} = \mathrm{Lan}_{\mathbf{k}}\mathbf{a}$), with unit η because η is the universal arrow to $\mathbf{X}^{\mathbf{k}}, \mathbf{b}$ from $\mathbf{a}$. Note that $\mathbf{X}^{\mathbf{k}} : \mathbf{X}^{\mathbf{A}} \longleftarrow \mathbf{X}^{\mathbf{B}}$ is defined by $\mathbf{X}^{\mathbf{k}}‘[\,] = [\,] \circ \mathbf{k}$ so $\mathbf{X}^{\mathbf{k}}‘\mathbf{b} = \mathbf{b} \circ \mathbf{k}$

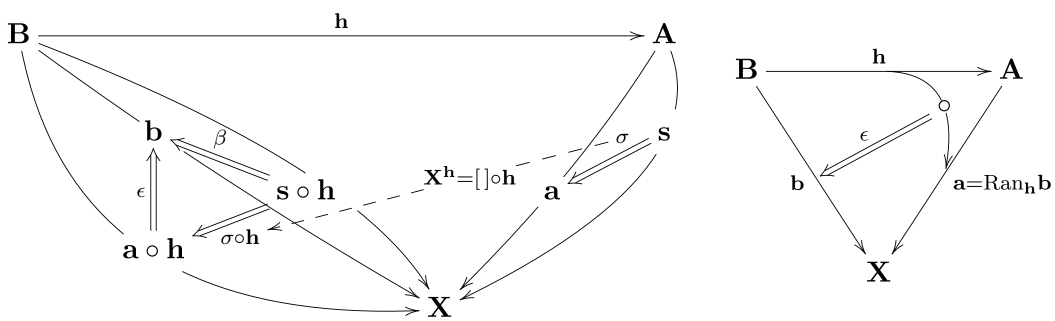


Figure 13: $\mathbf{a} : \mathbf{X} \longleftarrow \mathbf{A}$ is the right Kan extension of $\mathbf{b} : \mathbf{X} \longleftarrow \mathbf{B}$ along $\mathbf{h} : \mathbf{A} \longleftarrow \mathbf{B}$ (that is $\mathbf{a} = \mathrm{Ran}_{\mathbf{h}}\mathbf{b}$), with counit ϵ because ϵ is the universal arrow to $\mathbf{b}$ from $\mathbf{X}^{\mathbf{h}}, \mathbf{a}$. Note that $\mathbf{X}^{\mathbf{h}} : \mathbf{X}^{\mathbf{B}} \longleftarrow \mathbf{X}^{\mathbf{A}}$ is defined by $\mathbf{X}^{\mathbf{h}}‘[\,] = [\,] \circ \mathbf{h}$ so $\mathbf{X}^{\mathbf{h}}‘\mathbf{a} = \mathbf{a} \circ \mathbf{h}$

See figure 12. The left Kan extension along $\mathbf{k} : \mathbf{B} \longleftarrow \mathbf{A}$ is a canonical[10] way to extend the domain of a functor $\mathbf{a} : \mathbf{X} \longleftarrow \mathbf{A}$ to $\mathbf{B}$ so that it becomes a functor $\mathrm{Lan}_{\mathbf{k}}\mathbf{a} : \mathbf{X} \longleftarrow \mathbf{B}$.

It is an extension of $\mathbf{a}$ in the sense that $\mathrm{Lan}_{\mathbf{k}}\mathbf{a} \circ \mathbf{k} = \mathbf{a}$, it is canonical in the sense that it is the free such extension with respect to the grounding functor $\mathbf{X}^{\mathbf{k}}$. That is, there is a universal arrow to $\mathbf{X}^{\mathbf{k}}, \mathbf{b}$ from $\mathbf{a}$. Compare the double arrows in figure 12, with the definition of a free object in figure 9. If every functor has a left Kan extension, they together form a left adjoint (hence the name *left* Kan). Cf. Figure 10.

In the other direction, assume $\mathbf{h} : \mathbf{A} \longleftarrow \mathbf{B}$ so that $\mathbf{X}^{\mathbf{h}} = \lambda\mathbf{s}\,\textbf{.}\,\mathbf{s} \circ \mathbf{h} = [\,] \circ \mathbf{h} : \mathbf{X}^{\mathbf{B}} \longleftarrow \mathbf{X}^{\mathbf{A}}$.

See figure 13. The right Kan extension of a functor $\mathbf{b} : \mathbf{X} \longleftarrow \mathbf{B}$ along a functor $\mathbf{h} : \mathbf{A} \longleftarrow \mathbf{B}$ is determined by a universal arrow to $\mathbf{b}$ from $\mathbf{X}^{\mathbf{h}}, \mathbf{a}$ in the functor category $\mathbf{X}^{\mathbf{B}}$.

3.1.3.2 Kan Lift – Left and Right A lift is a dual, in a sense, of an extension. More specifically, see figrures 14 and 15

[10]The word "canonical" just means that, given $\mathbf{a}$ and $\mathbf{k}$, the definition of $\mathrm{Lan}_{\mathbf{k}}\mathbf{a}$ is fixed. It need not exist and when it does exist it need not be the only extension. Of course. all *Kan* extensions are isomorphic.

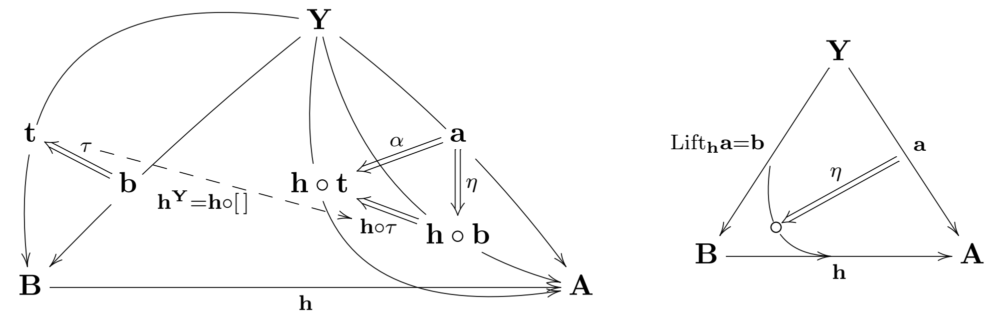


Figure 14: $\mathbf{b} : \mathbf{B} \longleftarrow \mathbf{Y}$ is the left Kan lift of $\mathbf{a} : \mathbf{A} \longleftarrow \mathbf{Y}$ along $\mathbf{h} : \mathbf{A} \longleftarrow \mathbf{B}$ (that is $\mathbf{b} = \mathrm{Lift}_{\mathbf{h}}\mathbf{a}$), with unit η because η is the universal arrow to $\mathbf{h}^{\mathbf{Y}}, \mathbf{b}$ from $\mathbf{a}$. Note that $\mathbf{h}^{\mathbf{Y}} : \mathbf{A}^{\mathbf{Y}} \longleftarrow \mathbf{B}^{\mathbf{Y}}$ is defined by $\mathbf{h}^{\mathbf{Y}}\text{‘}[\,] = \mathbf{h} \circ [\,]$ so $\mathbf{h}^{\mathbf{Y}}\text{‘}\mathbf{b} = \mathbf{h} \circ \mathbf{b}$

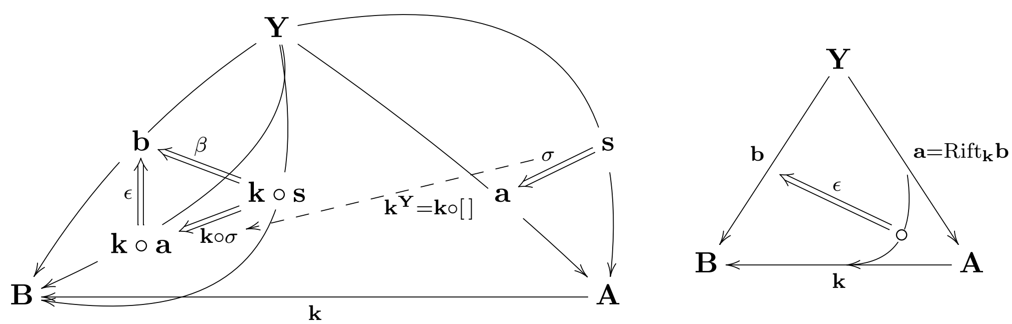


Figure 15: $\mathrm{Rift}_{\mathbf{k}}\mathbf{a} : \mathbf{B} \longleftarrow \mathbf{Y}$ is the right Kan lift of $\mathbf{a} : \mathbf{A} \longleftarrow \mathbf{Y}$ along $\mathbf{k} : \mathbf{A} \longleftarrow \mathbf{B}$ (that is $\mathbf{b} = \mathrm{Lift}_{\mathbf{h}}\mathbf{a}$), with co-unit η because η is the universal arrow to $\mathbf{h}^{\mathbf{Y}}, \mathbf{b}$ from $\mathbf{a}$. Note that $\mathbf{h}^{\mathbf{Y}} : \mathbf{A}^{\mathbf{Y}} \longleftarrow \mathbf{B}^{\mathbf{Y}}$ is defined by $\mathbf{h}^{\mathbf{Y}}\text{‘}[\,] = \mathbf{h} \circ [\,]$ so $\mathbf{h}^{\mathbf{Y}}\text{‘}\mathbf{b} = \mathbf{h} \circ \mathbf{b}$

3.1.3.3 Kan — Definitions

Definition 3.2. *Given two functors* $\mathbf{k} : \mathbf{B} \longleftarrow \mathbf{A}$*, and* $\mathbf{a} : \mathbf{X} \longleftarrow \mathbf{A}$*, a functor* $\mathbf{b} : \mathbf{X} \longleftarrow \mathbf{B}$*. is a left Kan extension, of* $\mathbf{a}$ *along* $\mathbf{k}$ *(in symbols:* $\mathbf{b} = \mathrm{Lan}_{\mathbf{k}}\mathbf{a}$*) iff there is a natural tranformation* $\eta :: \mathbf{X}^{\mathbf{k}}\text{‘}\mathbf{b} \Longleftarrow \mathbf{a} : \mathbf{X} \longleftarrow \mathbf{A}$ *which is a universal arrow to* $(\mathbf{X}^{\mathbf{k}}, \mathbf{b})$ *from* $\mathbf{a}$ *in the functor category* $\mathbf{X}^{\mathbf{A}}$.

Contrarywise, given two functors $\mathbf{h} : \mathbf{A} \longleftarrow \mathbf{B}$*, and* $\mathbf{b} : \mathbf{X} \longleftarrow \mathbf{B}$*, a functor* $\mathbf{a} : \mathbf{X} \longleftarrow \mathbf{A}$*. is a right Kan extension of* $\mathbf{b}$ *along* $\mathbf{h}$ *(in symbols:* $\mathbf{a} = \mathrm{Ran}_{\mathbf{h}}\mathbf{b}$*) iff there is a natural tranformation* $\epsilon :: \mathbf{b} \Longleftarrow \mathbf{X}^{\mathbf{h}}\text{‘}\mathbf{a} : \mathbf{X} \longleftarrow \mathbf{B}$ *which is a universal arrow to* $\mathbf{b}$ *from* $(\mathbf{X}^{\mathbf{h}}, \mathbf{a})$ *in the functor category* $\mathbf{X}^{\mathbf{B}}$.

Definition 3.3.

If $\mathbf{X} = \mathbf{B}$ then $\mathrm{Ran}_{\mathbf{k}}[\mathbf{B}] \dashv \mathbf{k}$ and $\mathrm{Ran}_{\mathbf{k}}\mathbf{b} = \mathbf{b} \circ \mathrm{Ran}_{\mathbf{k}}[\mathbf{B}]$

Every functor $\mathbf{t} : \mathbf{X} \longleftarrow \mathbf{B}$ has a right Kan extension along the functor $\mathbf{k} : \mathbf{A} \longleftarrow \mathbf{B}$ if and only if $\mathbf{X}^{\mathbf{k}} \dashv \mathrm{Ran}_{\mathbf{k}}$. Is this right? $(gf) = f \circ g$?

$g = \mathrm{Lan}_f(gf)$ ...`https://ncatlab.org/nlab/show/Kan+lift`

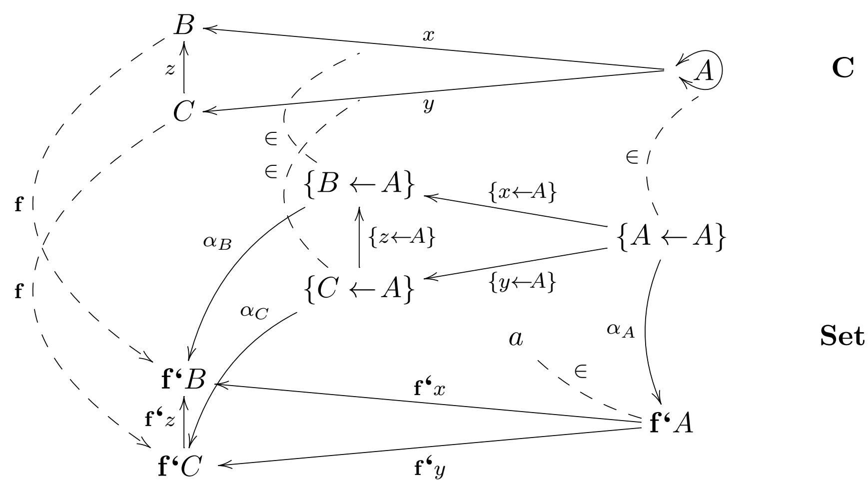


Figure 16: Yoneda

3.1.4 Yoneda and Co-Yoneda

Lemma 3.4 (Yoneda)**.** *If* $\mathbf{f} : \mathbf{Set} \longleftarrow \mathbf{C}$ *is a functor and* $A \in \mathbf{C}$ *on object, there is a bijection*

$$\phi_{\mathbf{f},A} : \mathbf{Set}^{\mathbf{C}}\{\mathbf{f} \Leftarrow \mathbf{C}\{[\,] \leftarrow A\}\} \longleftrightarrow \mathbf{f}‘A \text{ (in } \mathbf{Set})$$

Furthermore, these bijections form the components of a natural transformation (they form an isotransformation)

$$\phi :: \mathbf{nat} \Longleftrightarrow \mathbf{apply} : \mathbf{Set} \longleftarrow (\mathbf{Set}^{\mathbf{C}}) \times \mathbf{C}$$

where $\mathbf{apply}‘(\mathbf{f}, A) = \mathbf{f}‘A$ *and* $\mathbf{nat}‘(\mathbf{f}, A) = \{\mathbf{f} \Leftarrow \{[\,] \leftarrow A\}\}$.

Corollary 3.5 (Yoneda embedding)**.** *There is a full and faithful functor, called the Yoneda embedding:*

Let $\mathbf{f} = \mathbf{C}\{[\,] \leftarrow B\}$. *There is a bijection between* $\mathbf{Set}^{\mathbf{C}}\{\mathbf{C}\{[\,] \leftarrow B\} \Leftarrow \mathbf{C}\{[\,] \leftarrow A\}\}$ *and* $\mathbf{f}‘A = \mathbf{C}\{A \leftarrow B\}$.

Define the Yoneda embedding in each version and show that the lemma is equivalent to the proposition that it is full and faithful.

Proof. In particular, letting $\mathbf{f} = \mathbf{y}_{\mathbf{C}}^{+[\,]}$ so $\mathbf{f}‘A = \mathbf{y}_{\mathbf{C}}^{+A}$ or some better name, maybe $\mathbf{C}\!\upharpoonright_A^+$ prove the embedding. □

Recall the Natural End formula, Theorem 1.7: given two categories $\mathbf{C}$ and $\mathbf{D}$ with two functors $\mathbf{f}, \mathbf{g} : \mathbf{D} \longleftarrow \mathbf{C}$, the set of natural transformations between them can be written as the following end.

$$\mathbf{D}^{\mathbf{C}}\{\mathbf{g} \Leftarrow \mathbf{f}\} \cong \int_{C \in \mathbf{C}} \mathbf{D}\{\mathbf{g}‘C \leftarrow \mathbf{f}‘C\}$$

The next lemma is variously called the Density formula, the co-Yoneda lemma, the ninja[11] Yoneda lemma, or even just the Yoneda lemma. Just how do these correspond to the Yoneda lemma? Are two of them Yoneda and two co-Yoneda (density)? Just how is co-Yoneda dual to Yoneda? The Yoneda lemma is used twice in the proof, but does that make it "the Yoneda lemma in disguise"? Can the ordinay Yonda lemma be derived from the following?

Lemma 3.6 (ninja Yoneda)**.** *For any functors* $\mathbf{h} : \mathbf{C} \to \mathbf{Set}$ *and* $\mathbf{k} : \mathbf{C}^{op} \to \mathbf{Set}$

$$(i)\quad \mathbf{h} \;\cong\; \int^{C} \mathbf{h}`C \times \mathbf{C}\{C \leftarrow [\,]\} \qquad\qquad (ii)\quad \mathbf{h} \;\cong\; \int_{C} \mathbf{h}`C \,\hat{}\, \mathbf{C}\{[\,] \leftarrow C\}$$

$$(iii)\quad \mathbf{k} \;\cong\; \int^{C} \mathbf{k}`C \times \mathbf{C}\{[\,] \leftarrow C\} \qquad\qquad (iv)\quad \mathbf{k} \;\cong\; \int_{C} \mathbf{k}`C \,\hat{}\, \mathbf{C}\{C \leftarrow [\,]\}$$

Proof. The equivalences in the statement of the theorem are natural equivalences in the category of $\mathbf{Set}$ valued functors on $\mathbf{C}$. That category is composed of natual transformations between the objects which are such functors. They are proven by a series of isomorphisms,

So, to prove (1) let S be a set. Then

$$\begin{aligned}
\mathbf{Set}\left\{S \leftarrow \int^{C} \mathbf{h}`C \times \mathbf{C}\{C \leftarrow A\}\right\} &\cong \int_{C} \mathbf{Set}\Big\{S \leftarrow \mathbf{h}`C \times \mathbf{C}\{C \leftarrow A\}\Big\}\\
&\cong \int_{C} \mathbf{Set}\Big\{S^{\mathbf{h}`C} \leftarrow \mathbf{C}\{C \leftarrow A\}\Big\}\\
&\cong \mathbf{Set}^{\mathbf{C}}\{S^{\mathbf{h}`[\,]} \Leftarrow \mathbf{C}\{[\,] \leftarrow A\}\}\\
&\cong S^{\mathbf{h}`A} = \mathbf{Set}\big\{S \leftarrow \mathbf{h}`A\big\}
\end{aligned}$$

The first isomorphism is because the hom functor preserves co-ends (Loregian Cor. 1.2.8, p.15; CWM IX.5(4,5), p.221). The second is the exponential transpose. The third is Theorem 1.7, the above equivalence of an end with the set of natural transformations between functors $\mathbf{f}, \mathbf{g} : \mathbf{D} \longleftarrow \mathbf{C}$ where $\mathbf{D} = \mathbf{Set}$, $\mathbf{f} = \mathbf{C}\{[\,] \leftarrow A\}$, and $\mathbf{g} = S^{\mathbf{h}`[\,]}$. The fourth and final isomorphism is the ordinary Yoneda Lemma applied to the functor $S^{\mathbf{h}`[\,]} : \mathbf{Set} \longleftarrow \mathbf{C}$ and object $A \in \mathbf{Set}$.

Now isomorphisms in $\mathbf{Set}$ are just bijections, which exist between any two equinumerous sets, so this would be a weak claim, but these isomorphisms are natural in both S and A (right?).

To complete the proof we need to show that if $\mathbf{Set}\big\{S \leftarrow \mathbf{h_2}`A\big\} \cong \mathbf{Set}\big\{S \leftarrow \mathbf{h}`A\big\}$ for all A and naturally in S, then $\mathbf{h_2} \cong \mathbf{h}$. When that is proven just take $\mathbf{h_2} = \int^{C} \mathbf{h}`C \times \mathbf{C}\{C \leftarrow [\,]\}$.

Loregian [17, Prop.2.2.1(proof), p.35] says that parts (ii) and (iv) follow directly from the Natural End formula. Indeed, $\int_{C} \mathbf{h}`C \,\hat{}\, \mathbf{C}\{[\,] \leftarrow C\} \cong \mathbf{Set}^{\mathbf{C}}\{\mathbf{h} \Leftarrow \mathbf{C}\{[\,] \leftarrow C\}\} \cong \mathbf{h}`C$ □

Corollary 3.7 (op-Yoneda)**.** *If* $\mathbf{g} : \mathbf{Set} \longleftarrow \mathbf{C}^{op}$ *is a functor and* $B \in \mathbf{C}^{op}$ *there is a natural bijection*

$$\phi_{\mathbf{g},B} : \mathbf{Set}^{\mathbf{C}^{op}}\{\mathbf{g} \Leftarrow \mathbf{C}\{B \leftarrow [\,]\}\} \longleftrightarrow \mathbf{g}`B \text{ (in } \mathbf{Set})$$

[11] Loregian says:

"The name ninja Yoneda lemma is chosen [bcause it] is the Yoneda lemma in disguise." [17, Rem.2.2.2, p35] He blames the name on T. Leinster and cites `https://mathoverflow.net/q/20451`

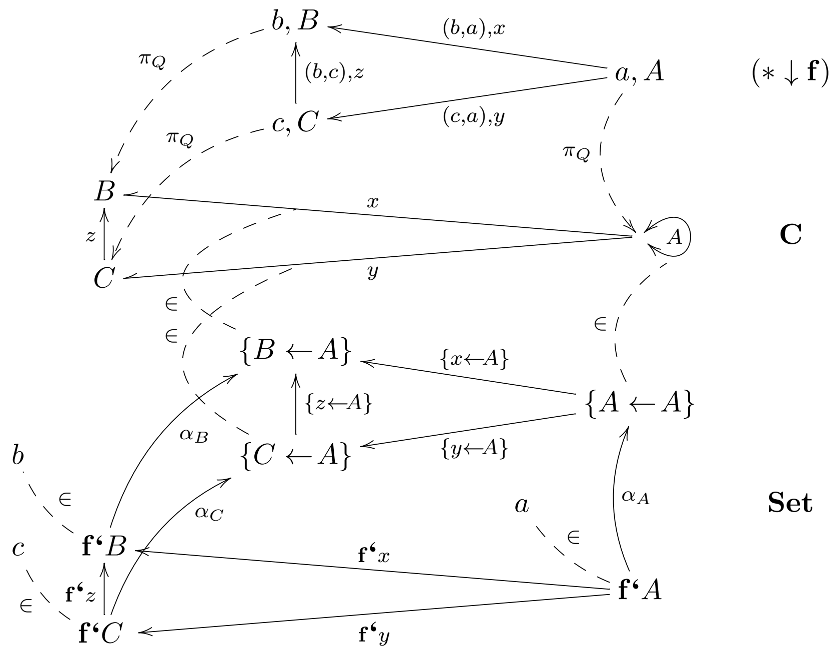


Figure 17: Co-Yoneda

Note that the natural transformation, α, goes in the opposite direction from that in the Yoneda lemma.

Given $\mathbf{k} : \mathbf{Set} \longleftarrow \mathbf{D}$, the category of elements is $(* \downarrow \mathbf{k})$ (See[12]). The category of elements is also denoted $\mathbf{D} \int \mathbf{k}$. (See[13]) Its objects are pairs $(C, p) \in \mathbf{C} \times P‘C$. —(What is P?)— Its arrows are $(u, Eq) : (C, p) \longleftarrow (C', p')$ where $u : C \longleftarrow C'$ and Eq is a proof that $p' =_{\mathbf{C}'} (P‘u)‘p$. This definition is part of the proof of Proposition 1 of Mac Lane and Moerdijk [19, §I.5],

Johnstone [10, Lem.0.12] says: "Any object of $\mathbf{Set}\,\hat{}\,\mathbf{C}^{op}$ can be expressed as the colimit of a diagram whose vertices are representable functors."

Bell [3, §1.10,p24] says: "any $F \in \mathbf{Set}^{\mathbf{C}^{op}}$ is the colimit of representable functors".

Kapulkin and Lumsdaine [12, Lem.2.1.5] say: "every simplicial set is canonically a colimit of standard simplices." What is a standard simplex, and in what category is the colimit taken? A standard topological simplex is the convex closure of basis vectors in Euclidian space. Riehl [22, p.4] says that the standard simplexes are the represented functors, $\Delta^n := \mathbf{\Delta}\{[n] \leftarrow []\}$.

Riehl [22, p.5] also says: "The density theorem, dual in some sense to the Yoneda lemma, says that any simplicial set is a colimit of standard simplices indexed by its category of elements $\varinjlim_{x \in X_n} \Delta^n \cong X$." (See §**??**, page **??** of these notes.)

Mac Lane says that a subcategory $\mathbf{M}$ of $\mathbf{C}$ is <u>dense</u> in $\mathbf{C}$ iff every object in $\mathbf{C}$ is a colimit of objects of $\mathbf{M}$ [18, CWM §IX.6, p.241].

[12] This definition is part of the coyoneda lemma Exercise 3, CWM [18, §III.2, p.62]

[13] This notation for the category of elements is used by Loregian [17, Def.A.6.11] who attibutes it to Gray [6].

Show there is a natural bijection $\widehat{\mathbf{C}}\{\mathbf{g}_{\geqslant}\varinjlim y \circ p\} \cong \{\Delta_{\mathbf{g}} \leftarrow \mathbf{f}\}$ where p is the projection from the category of elements $p\colon \mathbf{C} \longleftarrow \int_{\mathbf{C}} \mathbf{f}$. (Where did that come from? It's the Mac Lane and Moerdijk notation for the category of elements [19, p.43].

Is $\text{И}S, \forall A\,.\, P(S,A) \Longleftrightarrow \forall A,\ \text{И}S\,.\, P(S,A)$ valid ?

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